

OPERATIONAL RESEARCH

Quick Exam Cheat Sheet — All 7 Units

CAOR451 | BCA IV/VIII | 3 Credit Hrs

How to use this sheet

- ✓ Skim the boxed FORMULAS first — these carry the most marks.
- ✓ Green boxes = step-by-step method sequence (write these as your answer structure).
- ✓ Blue boxes = key formula / definition to memorize.
- ✓ Yellow boxes = fast recall bullets before you walk into the exam hall.

UNIT 1 — Introduction to Operations Research (5 Hrs)

Definition: OR is the application of scientific, mathematical methods to decision-making and resource allocation, aimed at finding the optimum solution to complex problems.

History & Stages of Development

- History: Originated in WWII (Britain, 1940s) to allocate scarce military resources; grew into industry/business post-1950.

Stages of OR Study

1. Formulating the problem (define objective, constraints, variables)
2. Constructing a mathematical model
3. Deriving a solution from the model
4. Testing the model and the solution (validation)
5. Establishing controls over the solution
6. Implementation of the solution

Manager ↔ OR Specialist Relationship

- Manager supplies the real-world problem, judgement, and final decision authority.
- OR specialist supplies the mathematical model, analysis, and optimum recommendation.
- Success needs close, continuous collaboration — model must reflect ground reality.

OR Tools & Techniques (recall list)

Technique	Used for
Linear Programming	Optimal allocation of limited resources
Transportation Model	Least-cost distribution from sources to destinations
Assignment Model	One-to-one job/machine allocation
Queuing Theory	Waiting-line / service system analysis
Inventory Models	Ordering & stock-level decisions

Technique	Used for
Replacement Theory	When to replace ageing equipment
Game Theory	Decision-making under competition

Applications & Limitations — Fast Recall

- ✓ Applications: production planning, finance, marketing, HR, logistics, defence.
- ✓ Limitation 1: Magnitude of computation is high for large real problems.
- ✓ Limitation 2: Quantitative basis may ignore qualitative/human factors.
- ✓ Limitation 3: Models are only as good as assumptions — gap between model & reality.
- ✓ Limitation 4: Implementation/communication gap between OR team & management.

UNIT 2 — Linear Programming Problem (10 Hrs)

LP Formulation — 3 Requirements

- Objective function: Maximize/Minimize $Z = c_1x_1 + c_2x_2 + \dots + c_nx_n$
- Constraints: $a_1x_1 + a_2x_2 \dots (\leq, =, \text{ or } \geq) b$
- Non-negativity: $x_1, x_2, \dots \geq 0$

Standard LP Form

$$\text{Max/Min } Z = \sum c_j x_j$$

Subject to: $\sum a_{ij}x_j (\leq / = / \geq) b_i$ for each constraint i
and $x_j \geq 0$ for all j

Graphical Method (2-variable problems)

Graphical Solution Steps

1. Plot each constraint as a straight line (treat inequality as equality to plot).
2. Shade the feasible region satisfying all constraints + non-negativity.
3. Identify all corner points (vertices) of the feasible region.
4. Evaluate Z at every corner point.
5. The corner point giving max/min Z is the optimal solution (Extreme Point Theorem).

Special Cases in Graphical Method

- ✓ Multiple Optimal Solutions: objective line parallel to a binding constraint — whole edge is optimal.
- ✓ Unbounded Solution: feasible region extends to infinity in the direction of improving Z .
- ✓ Infeasible Solution: no region satisfies all constraints simultaneously.

Simplex Method — Core Idea

- Used when more than 2 decision variables are involved.
- Convert inequalities to equalities using slack ($\leq$), surplus ($\geq$), and artificial variables.

- Start at origin (initial Basic Feasible Solution) and move along edges of the feasible region to an adjacent vertex that improves Z, until no further improvement is possible.

Simplex Method — Computation Steps

1. Standardize the LP: add slack/surplus/artificial variables to form equalities.
2. Set up the initial Simplex Table with the starting basic feasible solution.
3. Compute Z_j row and $(C_j - Z_j)$ row (index row).
4. Check optimality: for MAX, stop if all $(C_j - Z_j) \leq 0$.
5. If not optimal, select entering variable = most positive $(C_j - Z_j)$ column (key column).
6. Find leaving variable using Minimum Ratio Test = $\min(\text{RHS} \div \text{key column value, value} > 0) \rightarrow \text{key row}$.
7. Pivot: make key element = 1, reduce all other rows in key column to 0 (Gauss elimination).
8. Repeat steps 3–7 until optimal solution reached; read solution from RHS column.

Minimum Ratio Test (deciding leaving variable)

Ratio = (Solution / RHS value) $\div$ (positive entry in key column)

Leaving variable = row with the smallest non-negative ratio

Primal–Dual Relationship

Primal	Dual
Maximize Z	Minimize Z^*
m constraints, n variables	n constraints, m variables
$\leq$ type constraints	Variables ≥ 0
Coefficient of x_j in objective	RHS of jth dual constraint
RHS of ith constraint	Coefficient of y_i in dual objective

Note: Economic interpretation: the dual variable y_i (shadow price) = the change in optimal Z per unit increase in the RHS (availability) of resource i.

UNIT 3 — Transportation & Assignment Problem (8 Hrs)

Transportation Problem — Basics

- Goal: minimize total shipping cost from m sources to n destinations subject to supply and demand constraints.
- Balanced problem: Total supply = Total demand. If not, add a dummy row/column with cost 0 to balance it.
- Basic feasible solution needs exactly $m + n - 1$ occupied (allocated) cells.

Initial Basic Feasible Solution — 3 Methods

Method	Rule
North-West Corner (NWCN)	Allocate max possible at the top-left cell first, then move right/down; ignores cost — fastest but least optimal.

Method	Rule
Least Cost Method (LCM)	Allocate max possible to the cell with lowest cost first; repeat with next-lowest available cost.
Vogel's Approximation (VAM)	For each row/col find Penalty = (2nd lowest – lowest cost). Pick row/col with highest penalty; allocate max to its lowest-cost cell. Best starting solution, closest to optimal.

VAM Steps (most exam-favoured method)

1. Compute row and column penalties (difference between two lowest costs).
2. Select the row/column with the largest penalty.
3. Allocate as much as possible to the lowest-cost cell in that row/column.
4. Delete the satisfied row or column; recompute penalties for the rest.
5. Repeat until all supply/demand is allocated.

Optimality Test — MODI (Modified Distribution) Method

MODI Method Steps

1. Start from an initial BFS with $m+n-1$ allocations (use VAM ideally).
2. Compute u_i, v_j using: $C_{ij} = u_i + v_j$ for all OCCUPIED cells (set $u_1 = 0$ to start).
3. For each UNOCCUPIED cell compute penalty: $\Delta_{ij} = C_{ij} - (u_i + v_j)$.
4. If all $\Delta_{ij} \geq 0 \rightarrow$ solution is OPTIMAL. If any $\Delta_{ij} < 0 \rightarrow$ not optimal, proceed.
5. Choose most negative Δ_{ij} ; draw a closed loop from that cell through occupied cells.
6. Assign $+/-$ alternately around loop; shift = min of the '-' cells; update allocations.
7. Repeat until all $\Delta_{ij} \geq 0$.

Note: Degeneracy: occurs when occupied cells $< m+n-1$. Fix by allocating an infinitesimally small quantity ϵ to an unoccupied, independent cell.

- Unbalanced problems: add dummy source/destination with zero cost.
- Profit maximization: convert to a minimization problem by subtracting all profits from the maximum profit value, then solve as a normal minimization case.
- Transshipment problem: goods may pass through intermediate points before reaching final destination — solved as an expanded transportation problem where every node is both a source and destination.

Assignment Problem — Hungarian (Short-Cut) Method

Hungarian Method Steps

1. Row Reduction: subtract the minimum element of each row from every element in that row.
2. Column Reduction: subtract the minimum element of each column from every element in it.
3. Draw the minimum number of lines to cover all zeros. If lines = n (order of matrix) $\rightarrow$ optimal, go to step 5.
4. If lines $< n$: find smallest uncovered element; subtract it from all uncovered cells, add it to cells at line intersections; repeat step 3.
5. Make assignments: select rows/columns with a single zero, assign there, cross out row & column, repeat until all rows are assigned.

Case	Adjustment
Unbalanced ($n \neq m$ jobs/machines)	Add dummy row(s)/column(s) with zero cost
Maximization problem	Subtract every value from the matrix's maximum value, then minimize
Infeasible assignment	Assign a very large cost (M) to the forbidden cell
Crew assignment problem	Special assignment variant — minimize idle/waiting crew time between paired trips

UNIT 4 — Queuing Theory (6 Hrs)

Definition: Queuing (waiting-line) theory studies the behaviour of systems where customers arrive, wait if necessary, and are served — used to balance the cost of waiting against the cost of providing more service capacity.

Elements of a Queuing System

- Arrival process: pattern in which customers arrive (often Poisson).
- Service process: pattern/time taken to serve (often exponential).
- Number of servers / channels and their arrangement (series/parallel).
- Queue discipline: rule for picking next customer — commonly FCFS (First Come First Served); also LCFS, SIRO, priority.
- System capacity: finite or infinite waiting room.

Kendall's Notation: (a / b / c) : (d / e)

a = Arrival (inter-arrival time) distribution (M = Markovian/Poisson, D = Deterministic, G = General)

b = Service time distribution (M, D, G)

c = Number of parallel service channels (servers)

d = Queue discipline (FCFS, LCFS, SIRO, GD)

e = System capacity (N = finite, ∞ = infinite)

Operating Characteristics — Key Formulas (M/M/1 model)

M/M/1 : (FCFS / ∞) — single server, Poisson arrival λ , exponential service μ

Traffic intensity (utilization): $\rho = \lambda / \mu$ (must have $\rho < 1$ for steady state)

P(0 customers in system): $P_0 = 1 - \rho$

Expected number in system: $L_s = \lambda / (\mu - \lambda) = \rho / (1 - \rho)$

Expected number in queue: $L_q = \lambda^2 / [\mu(\mu - \lambda)] = \rho^2 / (1 - \rho)$

Expected waiting time in system: $W_s = 1 / (\mu - \lambda)$

Expected waiting time in queue: $W_q = \lambda / [\mu(\mu - \lambda)]$

Relationship (Little's Law): $L_s = \lambda \cdot W_s$ and $L_q = \lambda \cdot W_q$

Classification of Queuing Models

Model	Description
M/M/1 : (FCFS/ ∞/∞)	Single server, infinite population & capacity
M/M/c : (FCFS/ ∞/∞)	Multiple (c) parallel servers, same discipline
M/M/1 : (FCFS/N/ ∞)	Single server, finite system capacity N
M/M/1 : (FCFS/ ∞ /N)	Single server, finite calling population N

Queuing — Fast Recall

- ✓ λ = mean arrival rate; μ = mean service rate — always check units (per hour/min) match.
- ✓ Steady state exists only when $\lambda < \mu$ ($\rho < 1$); otherwise queue grows infinitely.
- ✓ Goal in exam numericals: compute ρ , P_0 , L_s , L_q , W_s , W_q in that order.

UNIT 5 — Inventory Control (6 Hrs)

Inventory Classification

- Raw materials, Work-in-progress (WIP), Finished goods, Spares/MRO items.
- Held to guard against demand/lead-time uncertainty and to gain economies of scale.

Costs Associated with Inventory

Cost Type	Meaning
Ordering / Set-up cost (C_o)	Cost incurred each time an order/production run is placed
Carrying / Holding cost (C_h)	Cost of storage, insurance, capital tied up per unit per year
Shortage / Stock-out cost	Cost of lost sales/goodwill when demand isn't met
Purchase cost	Actual price paid per unit of item

Economic Order Quantity (EOQ) — Wilson Formula

$$EOQ (Q^*) = \sqrt{(2DC_o / C_h)}$$

D = annual demand, C_o = ordering cost/order, C_h = holding cost/unit/year

Number of orders per year: $N = D / Q^*$

Time between orders: $t = Q^* / D$ (in year fraction) $\rightarrow \times 365$ for days

Total Annual Cost: $TC = (D/Q)C_o + (Q/2)C_h + (\text{Purchase cost} \times D)$

Note: At EOQ, Ordering Cost = Carrying Cost exactly — the two curves intersect at the minimum of the Total Cost curve.

Inventory Models with Deterministic Demand

- EOQ without shortages — basic model above; demand & lead time known and constant.
- EOQ with shortages allowed — adds shortage cost C_s ; $Q^* = \sqrt{(2DC_o/C_h)} \times \sqrt{((C_h+C_s)/C_s)}$
- EOQ with finite (gradual) replenishment — production rate $P >$ demand rate D ; $Q^* = \sqrt{(2DC_o/C_h)} \times \sqrt{(P/(P-D))}$

ABC Analysis

Class	% of Items	% of Annual Value	Control Level
A	~10%	~70%	Tight — frequent review, accurate records
B	~20%	~20%	Moderate — periodic review
C	~70%	~10%	Loose — bulk ordering, simple control

Inventory — Fast Recall

- ✓ EOQ minimizes Total Cost = Ordering Cost + Carrying Cost.
- ✓ ABC analysis is a selective inventory control technique — focus control effort where value is highest.
- ✓ Always check units before plugging into EOQ formula (annual demand, per-unit costs).

UNIT 6 — Replacement Theory (6 Hrs)

Definition: Replacement theory addresses WHEN to replace equipment/assets that deteriorate with time, wear out suddenly, or become technologically obsolete — balancing rising maintenance cost against depreciation and new investment cost.

Replacement of Items that Deteriorate Gradually (with time)

Optimal Replacement Age — Average Cost Method

Average Annual Cost, $AC(n) = [C - S(n) + \sum \text{Maintenance costs up to year } n] / n$

C = purchase cost, $S(n)$ = resale/scrap value after n years

Replace the equipment in the year n where $AC(n)$ is minimum, i.e. the year just before $AC(n)$ starts increasing again

Rule of thumb: replace when next year's running cost > current average cost $AC(n)$

Replacement with Time Value of Money (Present Worth)

Present Worth Factor Method

Discount factor for year n : $v = 1 / (1+r)$ at rate r ; PW factor for year $n = v^n$

Weighted Average Cost = $\sum(\text{Running Cost} \times \text{PW factor}) / \sum(\text{PW factor})$, added to purchase cost

Replace in the year where the discounted average annual cost is minimum

Replacement by Alternative Equipment

- Compare average annual cost (or discounted cost) of the OLD machine's remaining life against the average annual cost of the NEW/alternative machine over its own life.
- Choose whichever option has the lower equivalent annual cost — replace only if the new option is cheaper on a like-for-like annual basis.

Items that Fail Suddenly — Group vs Individual Replacement

Policy	When to Use	Rule
Individual Replacement	Failure cost is high per item, or items fail independently	Replace each item immediately upon its failure
Group Replacement	Large number of low-cost identical items (e.g. bulbs)	Replace ALL items at fixed intervals + replace individual failures as they occur in between; compare total cost of group policy vs. pure individual policy

Group Replacement — Decision Rule

Total Cost (group policy at interval n) = $n \cdot (\text{individual failure replacement cost of failures before } n) + N \cdot (\text{group replacement cost})$

Choose the group replacement interval n that MINIMIZES average total cost per unit time

Replacement Theory — Fast Recall

- ✓ Gradual deterioration → use Average Cost / Present Worth method, find minimum-cost year.
- ✓ Sudden failure, many cheap items → compare Group vs Individual replacement cost.
- ✓ Always replace when marginal (next year) cost exceeds current average cost.

UNIT 7 — Game Theory (7 Hrs)

Definition: Game theory analyzes decision-making in competitive situations where two or more players (with conflicting interests) choose strategies whose outcome (payoff) depends on the choices of all players.

Characteristics of a Game

- Finite number of competitors (players).
- Each player has a finite (or infinite) list of possible courses of action (strategies).
- A play occurs when each player selects one strategy; results in a payoff.
- Payoff (gain/loss) is fixed and known in advance for every combination of strategies.

Two-Person Zero-Sum Game

- Exactly two players; one player's gain = exactly the other player's loss (sum of payoffs = 0).
- Represented as a Payoff Matrix: rows = Player A's strategies, columns = Player B's strategies, values = A's payoff (gain).

Pure Strategy — Saddle Point Method

Steps to Solve (Pure Strategy)

1. Find the minimum value in each ROW → note the Row Minima.
2. Select the MAXIMUM of these row minima → Maximin value.
3. Find the maximum value in each COLUMN → note the Column Maxima.
4. Select the MINIMUM of these column maxima → Minimax value.
5. If Maximin = Minimax → SADDLE POINT exists → this is the value of the game (V); the corresponding row/column strategies are the optimal pure strategies.

Note: If Maximin $\neq$ Minimax, no saddle point exists $\rightarrow$ pure strategy fails, solve using mixed strategy or dominance.

Dominance Principle (Reducing the Matrix)

- For Player A (maximizing, rows): a row is dominated (deleted) if every element in it is $\leq$ the corresponding element of another row.
- For Player B (minimizing, columns): a column is dominated (deleted) if every element in it is $\geq$ the corresponding element of another column.
- Repeat elimination to reduce the matrix, then re-check for a saddle point or solve the smaller mixed-strategy matrix.

Mixed Strategy — 2×2 Games (Algebraic Method)

2×2 Payoff Matrix | a b | | c d |

Player A's probability for row 1: $p = (d - c) / [(a+d) - (b+c)]$, $p_2 = 1 - p$

Player B's probability for col 1: $q = (d - b) / [(a+d) - (b+c)]$, $q_2 = 1 - q$

Value of the Game: $V = (ad - bc) / [(a+d) - (b+c)]$

m × 2 or 2 × n Games — Graphical Method

Graphical Method Steps (2-strategy player)

1. Plot the payoffs of the 2-strategy player's two strategies as two parallel lines (x-axis = probability p, 0 to 1).
2. For maximizing player: identify the UPPER envelope (highest points) of the lines.
3. For minimizing player: identify the LOWER envelope (lowest points) of the lines.
4. The optimal point is where two lines forming the envelope INTERSECT — gives the value of the game and optimal mixed probability p.
5. Reduces the m×2 (or 2×n) game to an equivalent 2×2 game using the two intersecting strategies, then solve algebraically for the opponent's probabilities.

Game Theory — Fast Recall

- ✓ Always check for a saddle point FIRST (Maximin = Minimax) before trying mixed strategy.
- ✓ Apply Dominance to shrink big matrices before solving.
- ✓ 2×2 mixed strategy: memorize the $V = (ad-bc)/[(a+d)-(b+c)]$ formula — very high yield.
- ✓ Value of game $V > 0$ favors Player A (row); $V < 0$ favors Player B (column); $V = 0$ is a fair game.

— End of Cheat Sheet — All the Best! —