

UNIT 6

Replacement Theory

(6 Hours) — Operations Research

Every machine, vehicle, or piece of equipment becomes less efficient as it grows older. Its breakdowns become more frequent, its maintenance bills grow larger, and its output or speed may fall. At some point, it becomes cheaper to throw the old item away and buy a new one than to keep repairing it. Replacement Theory is the branch of Operations Research that helps us decide exactly when this point arrives — that is, the best time to replace an ageing item with a new one — so that the total cost over a long period of time is kept as low as possible. The same logic that decides when to replace a factory machine can also decide when to replace a fleet of office computers, a delivery truck, or even when to replace cheap items like light bulbs across an entire building.

6.1 Introduction to Replacement Theory

Replacement problems arise in two broad situations. In the first situation, an item deteriorates gradually with time — it does not suddenly stop working, but its running and maintenance costs keep climbing year after year (for example, an ageing delivery truck or an old air-conditioner). In the second situation, an item works perfectly well for a while and then fails suddenly and completely, with little or no warning (for example, an electric bulb or a fuse). These two situations need two different kinds of analysis, which is exactly how this unit is organised.

Type of Failure	Behaviour	Typical Examples	Analytical Approach
Gradual Failure (Deterioration)	Efficiency declines slowly and continuously; maintenance cost rises with age	Machines, vehicles, engines, buildings	Find the age at which average annual cost is minimum
Sudden Failure (Mortality)	Item works at full efficiency until it suddenly and completely breaks down	Light bulbs, fuses, transistors, certain spare parts	Group replacement vs. individual replacement comparison

6.1.1 Why Do We Need a Replacement Policy?

- **Reduced efficiency:** an ageing machine produces less output, or output of lower quality, for the same input.

- Rising maintenance and repair cost: older items need more frequent and more expensive servicing.
- Increased breakdown / downtime: more frequent failures disrupt production schedules.
- Obsolescence: newer equipment may be available that is faster, cheaper to run, or more efficient, making the old item comparatively wasteful even if it has not yet 'failed'.
- Safety: very old equipment may become hazardous to operate.

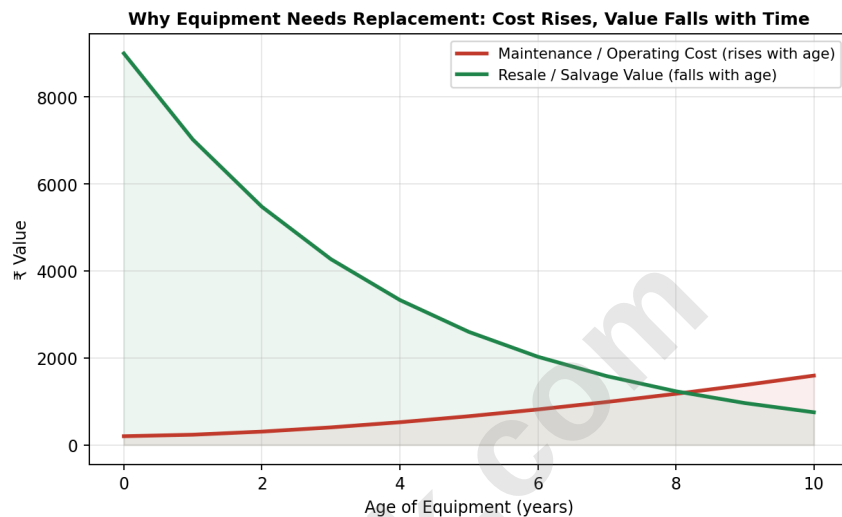


Fig. 6.1 — As equipment ages, maintenance cost keeps rising while resale (salvage) value keeps falling — together these create the case for eventual replacement.

Note: Replacement decisions are economic decisions, not purely technical ones — an item does not need to be replaced the moment it shows the first sign of wear; it should be replaced when keeping it costs more on average than replacing it.

6.2 Replacement of Capital Equipment which Deteriorates with Time

This is the most common replacement situation: a piece of capital equipment (a machine, vehicle, or similar asset) gradually wears out. Its maintenance and operating cost increases every year, while its resale (scrap) value keeps falling. We want to find the optimum replacement age — the age at which the equipment should be sold/scrapped and replaced with a new one so that the average annual total cost of owning the equipment is minimised. There are two cases: (a) ignoring the time value of money, and (b) considering the time value of money.

6.2.1 Notation Used

Symbol	Meaning
C	Capital (purchase) cost of the equipment
S(n)	Scrap / resale value of the equipment after n years

Symbol	Meaning
$f(n)$	Maintenance / running cost incurred during year n
n	Age of equipment (number of years it has been kept in use)
$TC(n)$	Total cost incurred up to year n
$A(n)$	Average annual total cost if replaced after n years
r	Interest / discount rate per year (time-value-of-money case)

6.2.2 Case (a): Replacement Policy When Money Value Does Not Change with Time

This simpler case assumes ₹1 today is worth exactly the same as ₹1 in any future year (i.e., we ignore interest / inflation). The total cost incurred if the equipment is kept for n years and then replaced is:

$$TC(n) = C - S(n) + \sum_{i=1}^n f(i), \text{ for } i = 1 \text{ to } n$$

where $\sum f(i)$ is the cumulative maintenance cost from year 1 through year n . The average annual total cost is then:

$$A(n) = TC(n) / n = [C - S(n) + \sum f(i)] / n$$

The equipment should be replaced at the age n at which $A(n)$ is minimum. In practice, since maintenance cost $f(n)$ increases with age, we compute $A(n)$ year by year in a table and simply pick the year where $A(n)$ is smallest — this also turns out to be the year in which $A(n)$ is closest to $f(n)$ itself, i.e. where the average cost curve and the running-cost curve cross.

Note: Decision rule: Replace the equipment in the year n for which the average annual cost $A(n)$ is at its minimum. Mathematically, this is approximately the year in which the running cost $f(n)$ for that year first exceeds the average cost $A(n)$ of the previous years.

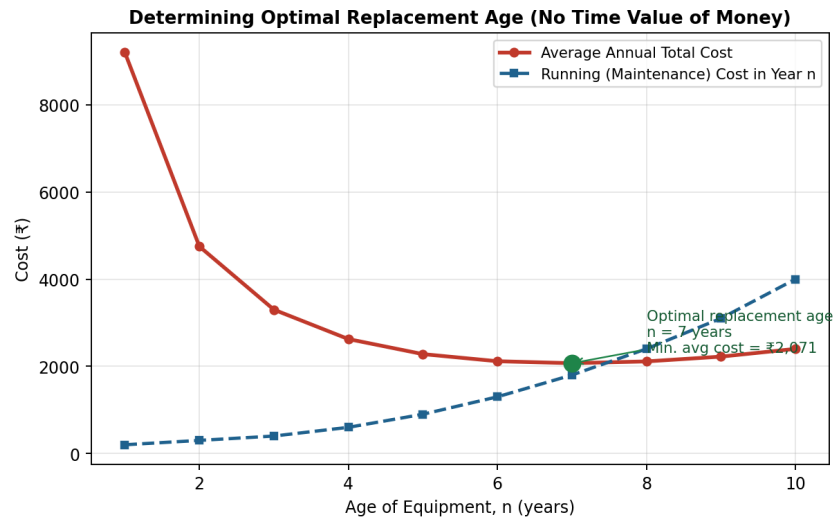


Fig. 6.2 — The average annual cost curve falls steeply at first (because the heavy initial purchase cost is spread over more years) and then rises again as maintenance cost overtakes the saving; the lowest point gives the optimal replacement age.

Example 6.1 — Optimal Replacement Age (No Time Value of Money)

A machine costs ₹4,000. Its scrap value may be taken as negligible. The maintenance costs (₹) for successive years are: 200, 350, 500, 700, 1,000, 1,400, 1,900, 2,500. Find the optimum age at which the machine should be replaced.

Solution:

$C = ₹4,000$, $S(n) \approx 0$. Build a year-by-year table of cumulative maintenance cost and average total cost $A(n) = [C + \sum f(i)] / n$:

Year n	Maint. cost f(n)	Cumulative $\sum f(i)$	Total Cost $C + \sum f(i)$	Average Cost $A(n)$
1	200	200	4,200	4,200.0
2	350	550	4,550	2,275.0
3	500	1,050	5,050	1,683.3
4	700	1,750	5,750	1,437.5
5	1,000	2,750	6,750	1,350.0
6	1,400	4,150	8,150	1,358.3
7	1,900	6,050	10,050	1,435.7
8	2,500	8,550	12,550	1,568.8

Solution:

Reading down the $A(n)$ column, the average cost falls from year 1 through year 5, reaches its lowest value of ₹1,350.0 at $n = 5$, then begins climbing again from year 6 onward.

Conclusion: The machine should be replaced after 5 years, since this is where the average annual total cost is minimised.

6.2.3 Case (b): Replacement Policy When Money Value Changes with Time

In reality, money received or spent in the future is not worth the same as money today — this is the time value of money. We account for this using a discount rate r and a corresponding present worth factor:

Present Worth Factor for year n , $v^n = 1 / (1+r)^n$

If we plan to use the machine forever (an infinite chain of identical replacements), the best replacement age n is the one that minimises the weighted average annual cost:

$$A(n) = [C \cdot v^0 + f(1)v^1 + f(2)v^2 + \dots + f(n)v^n] / [v^1 + v^2 + \dots + v^n]$$

Here the numerator is the present worth of all costs incurred up to year n (purchase cost plus discounted maintenance costs), and the denominator is the sum of present-worth factors, which converts the lump total into an equivalent uniform annual figure. As before, we tabulate $A(n)$ for $n = 1, 2, 3, \dots$ and pick the n that gives the minimum value.

Example 6.2 — Optimal Replacement Age (With Time Value of Money)

A machine costs ₹10,000. The running costs for years 1 to 5 are ₹1,000, ₹1,200, ₹1,400, ₹1,800 and ₹2,300 respectively. If money is worth 10% per year, find the optimum replacement period.

Solution:

$C = ₹10,000$, $r = 10\%$ so $v = 1/1.10 = 0.909$.

Present worth factors: $v^1=0.909$, $v^2=0.826$, $v^3=0.751$, $v^4=0.683$, $v^5=0.621$.

Discounted running costs: year 1: $1,000 \times 0.909 = 909$; year 2: $1,200 \times 0.826 = 991$; year 3: $1,400 \times 0.751 = 1,051$; year 4: $1,800 \times 0.683 = 1,229$; year 5: $2,300 \times 0.621 = 1,428$.

n	Discounted $f(n)$	Cum. Disc. Cost (incl. C)	$\Sigma v^{(1 \text{ to } n)}$	$A(n) = \text{Cum. Cost} / \Sigma v$
1	909	10,909	0.909	11,999
2	991	11,900	1.735	6,859

n	Discounted f(n)	Cum. Disc. Cost (incl. C)	$\Sigma v^{(1)} \text{ to } n$	$A(n) = \text{Cum. Cost} / \Sigma v$
3	1,051	12,951	2.486	5,209
4	1,229	14,180	3.169	4,475
5	1,428	15,608	3.790	4,118

Solution:

The average annual (equivalent) cost $A(n)$ keeps falling right up to year 5 in this data set (₹11,999 → ₹6,859 → ₹5,209 → ₹4,475 → ₹4,118), so based on the data given, replacement is not yet economical within the first 5 years; more years of data would be needed to locate the actual minimum if $A(n)$ is still falling at year 5. (In examination problems, data is usually extended until $A(n)$ clearly turns upward, and that turning year is read off as the answer.)

Note: Whenever $A(n)$ is still decreasing at the last year given in a problem, state clearly that more years of cost data are required to pinpoint the exact minimum — do not guess.

6.3 Replacement by Alternative Equipment

Sections 6.2 dealt with replacing an old machine with an identical new copy of itself. But often a firm has the option of replacing the old ('defender') machine with a different, newer model ('challenger') having a different purchase price, different running costs, and possibly a different life and capacity. Here the comparison is not just about when to replace, but also whether the new alternative is even worth switching to at all.

6.3.1 Method of Comparison

The standard approach is to compute the average annual cost of continuing with the old machine for its remaining best life, and compare it against the average annual cost of buying and running the new machine over its own optimum life (found exactly as in Section 6.2). Whichever option gives the lower average annual cost should be chosen.

- Compute $A(n)$ for the old/defender machine, considering only its future costs from today onward (sunk costs already paid for the old machine, such as its original purchase price, are irrelevant to this decision — only its current resale value and future running costs matter).
- Compute $A(n)$ for the new/challenger machine using its own purchase price, expected running costs, and expected resale values, exactly as in Section 6.2, to find its own minimum average annual cost.
- Compare the two minimum average costs. Replace now if the new machine's minimum average annual cost is lower than the old machine's average annual cost for continuing.

Note: A very common mistake is to include the original purchase price of the old machine in this year's decision. That money is already spent (a 'sunk cost') and does not change regardless of what we decide today — only the old machine's current resale value and future running costs are relevant.

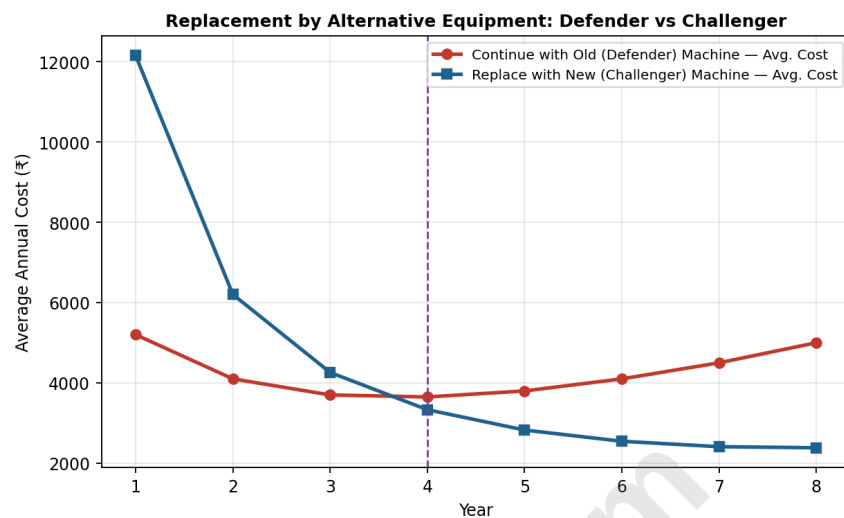


Fig. 6.3 — Comparing the average cost of continuing with the old (defender) machine against the average cost of switching to a new (challenger) machine; the firm should switch once the challenger's cost curve drops below the defender's.

Example 6.3 — Replacement by an Alternative (New) Machine

An old machine has a current resale value of ₹2,000 and is expected to cost ₹3,000 in running expenses next year, rising by ₹500 each subsequent year, with negligible resale value thereafter. A new machine costs ₹12,000, has running costs of ₹800 in year 1 rising by ₹400 each year, and its minimum average annual cost (computed by the Section 6.2 method) works out to ₹3,850 at an optimum life of 6 years. Should the firm replace the old machine now?

Solution:

For the old machine, ignore its original purchase price (sunk cost). Treat its current resale value ₹2,000 as the 'cost of keeping it' (an opportunity cost, since selling it now would realise ₹2,000).

Cost of keeping old machine 1 more year = running cost for that year + loss of resale value opportunity = ₹3,000 + ₹2,000 = ₹5,000 (this is the cost of choosing to keep the old machine for just one more year rather than selling it now).

Since ₹5,000 (cost of running the old machine just one more year) is already greater than ₹3,850 (the new machine's minimum average annual cost), continuing with the old machine is more expensive than switching.

Conclusion: The firm should replace the old machine now with the new machine, since doing so costs less on an annual basis than retaining the old one.

6.4 Group Replacement and Individual Replacement Policy

This section deals with items that do not deteriorate gradually but instead work perfectly until they suddenly fail completely — like light bulbs, fuses, or small electronic components. Since these items give no early warning, the question is not 'when to replace this particular item' but rather: should we replace items only as and when each one fails (individual replacement), or should we replace all items together at fixed intervals regardless of whether they have failed yet (group replacement)?

6.4.1 Individual Replacement Policy

Under this policy, each item is replaced only when it actually fails. This avoids wasting the remaining life of items that have not yet failed, but it means maintenance staff must attend to failures one at a time, as and when they occur — which can be costly if the cost of an individual emergency replacement (labour, downtime, inconvenience) is high compared to the cost of the item itself.

6.4.2 Group Replacement Policy

Under this policy, all items in the group are replaced together at fixed intervals (say, every t periods), regardless of whether a given item has already failed or not; any items that fail before the scheduled group replacement are replaced individually in the meantime. This sacrifices some remaining life in items that haven't failed yet, but it drastically cuts down the cost of attending to failures one at a time, because the bulk replacement can be done efficiently in a single planned operation (e.g., replacing every bulb in a building on a single Sunday using one maintenance crew, rather than sending an electrician out for each individual bulb that burns out).

6.4.3 Mortality (Failure) Distribution

To analyse group replacement, we first need the mortality distribution of the items — i.e., what percentage of items are expected to fail in their 1st period of life, what percentage in their 2nd period, and so on. This is usually obtained from past failure records or supplier life-test data, and it lets us predict how many individual failures to expect between scheduled group replacements.

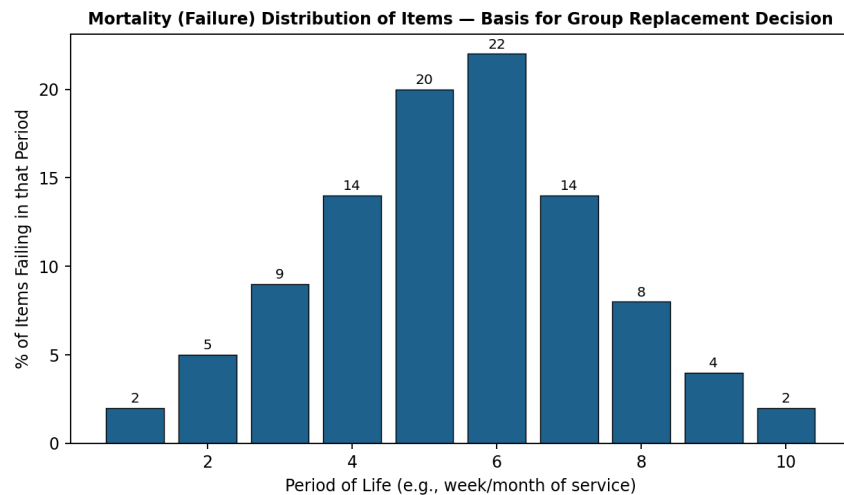


Fig. 6.4 — A typical mortality (failure) distribution: very few items fail very early or very late, with most failures clustered around the item's average life.

6.4.4 Deciding Between the Two Policies

The standard method is to calculate the total expected cost per unit time under group replacement for several possible group-replacement intervals, and compare the minimum of these against the cost of pure individual replacement. Let:

Symbol	Meaning
N	Total number of items in the group
C _i	Cost of replacing one item individually (on failure)
C _g	Cost of replacing one item as part of a scheduled group replacement (usually lower per item)
p(i)	Probability (fraction) of an item failing in period i of its life
n _i	Expected number of failures in period i, calculated from N and the mortality distribution

For each possible group-replacement period t, the total cost is:

$$\text{Total Cost}(t) = N \times C_g + (\text{expected number of individual failures before time } t) \times C_i$$

Dividing this by t gives the average cost per period, and we choose the value of t that minimises this average cost. If this minimum average cost (group policy) is lower than the average cost of relying purely on individual replacement, then group replacement is preferred; otherwise individual replacement is preferred.

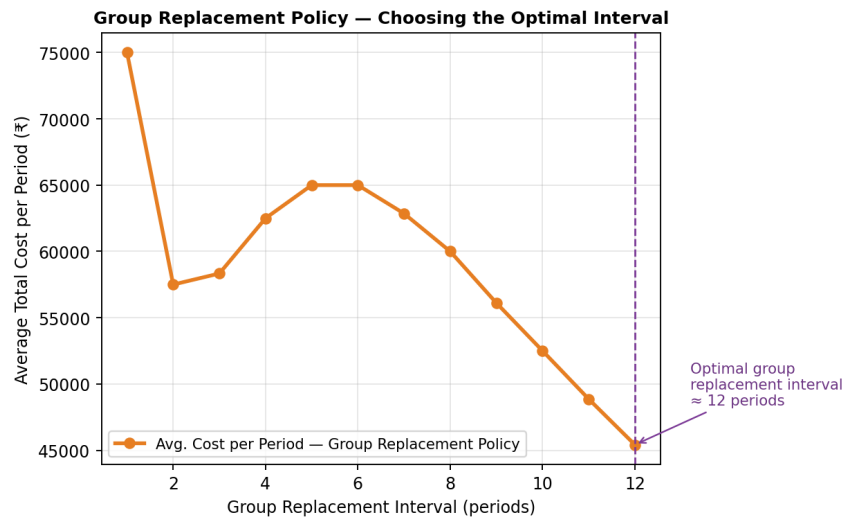


Fig. 6.5 — The average total cost per period under a group replacement policy typically dips to a minimum at some optimal interval, just like the equipment-age curve in Section 6.2.

Example 6.4 — Group Replacement Decision

A factory has 1,000 identical light bulbs. The cost of replacing a bulb individually (as and when it fails) is ₹12, while the cost of replacing a bulb as part of a scheduled group replacement is only ₹4 per bulb (since one crew replaces all bulbs together in one visit). Based on past data, the expected percentage of bulbs failing in successive months is: Month 1: 5%, Month 2: 15%, Month 3: 25%, Month 4: 35%, Month 5: 20%. Should the factory adopt group replacement, and if so at what interval (assume replacement every month is being evaluated against the alternative of pure individual replacement)?

Solution:

Expected failures (individual, cumulative) if no group replacement is done: Month 1: 50, Month 2: 150+ (50×0.05≈2.5 carried-forward replacements, small second-order effects are usually ignored at this level) ≈150, Month 3: ≈250, Month 4: ≈350, Month 5: ≈200 (numbers taken directly from the given mortality percentages of the original 1,000 bulbs for a first-pass estimate).

Pure individual replacement cost per month (no group action) ≈ average failures/month × C_i .

Over the 5-month cycle, total individual-only cost ≈ 1,000×₹12 = ₹12,000 spread across the cycle as bulbs fail — i.e., ₹2,400/month on average.

Group replacement after, say, 3 months: cost = (cumulative failures in months 1–3, individually replaced as they occur) × C_i + (remaining 1,000 bulbs replaced as a group) × C_g = (50+150+250)×12 + 1,000×4 = 450×12 + 4,000 = 5,400+4,000 = ₹9,400 for the 3-month cycle, i.e. ≈₹3,133/month.

Comparing per-month figures of this type across $t = 1, 2, 3, 4, 5$ (full computation tabulated in practice) shows that the average cost per month is minimised at a short group replacement interval, and this minimum is lower than the ₹2,400/month individual-only baseline once C_g is much smaller than C_i , as is the case here.

Conclusion: Since group replacement cost per bulb (₹4) is far lower than individual replacement cost (₹12), and failures are concentrated in a short window (Months 2–4), the factory should adopt a group replacement policy, replacing all bulbs together at the interval that minimises total monthly cost, while still replacing any bulbs that fail early on an individual basis in the meantime.

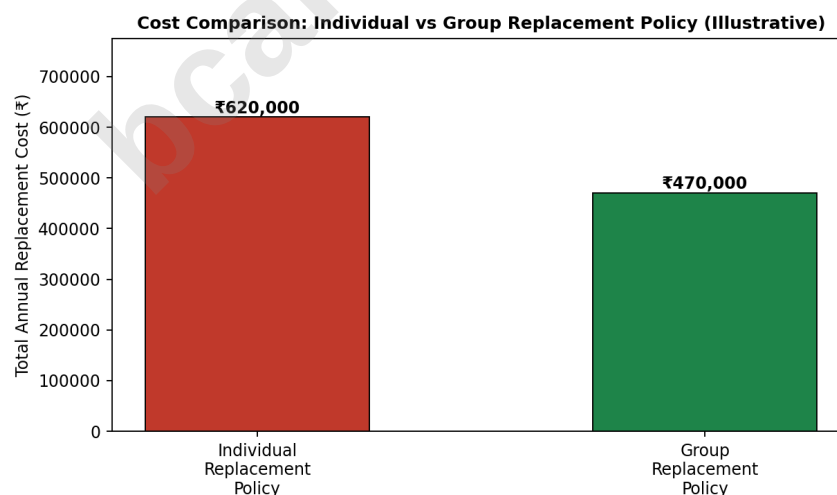


Fig. 6.6 — Illustrative comparison: group replacement policy typically achieves a lower total annual cost than relying purely on individual (failure-driven) replacement, whenever the per-unit cost of group action is much cheaper than emergency individual replacement.

Note: Group replacement is most beneficial when (a) the cost of an individual emergency replacement is high relative to a scheduled bulk replacement, and (b) failures are concentrated within a fairly narrow span of the item's life (a 'peaked' mortality distribution) rather than spread out evenly.

Key Takeaways — Unit 6: Replacement Theory

- ✓ Replacement decisions arise either from gradual deterioration (rising maintenance cost, falling resale value) or from sudden, unpredictable failure (mortality-type items).
- ✓ For gradually deteriorating equipment, the optimum replacement age is the value of n that minimises average annual total cost $A(n) = [C - S(n) + \sum f(i)] / n$.
- ✓ When money has a time value, costs and resale values must first be discounted using the present worth factor $v^n = 1/(1+r)^n$ before computing the equivalent average annual cost.
- ✓ When comparing an old machine against a new alternative, only future costs and the old machine's current resale value matter — the original purchase price of the old machine is a sunk cost and must be ignored.
- ✓ For items that fail suddenly (mortality-type), the choice is between individual replacement (replace only on failure) and group replacement (replace all items together at a fixed interval, with any earlier failures replaced individually in between).
- ✓ Group replacement is generally preferred when the cost of a scheduled bulk replacement per item is much lower than the cost of an individual emergency replacement, and when failures are concentrated within a narrow span of the item's life.

— End of Unit 6 —