

UNIT 7

Design of Experiment

(6 Hours) — Probability & Statistics (CACSS202)

7.1 Concept of Analysis of Variance (ANOVA)

A **t-test** compares the means of *two* groups. When we must compare the means of *three or more* groups — for example the yield of three fertilizers, or the marks obtained by students taught with four methods — repeated t-tests are unsuitable because every extra test increases the chance of a false conclusion (Type I error). **Analysis of Variance (ANOVA)**, developed by R. A. Fisher, tests the equality of several population means in a single test by *splitting the total variation in the data into parts*, each part being due to a recognised source.

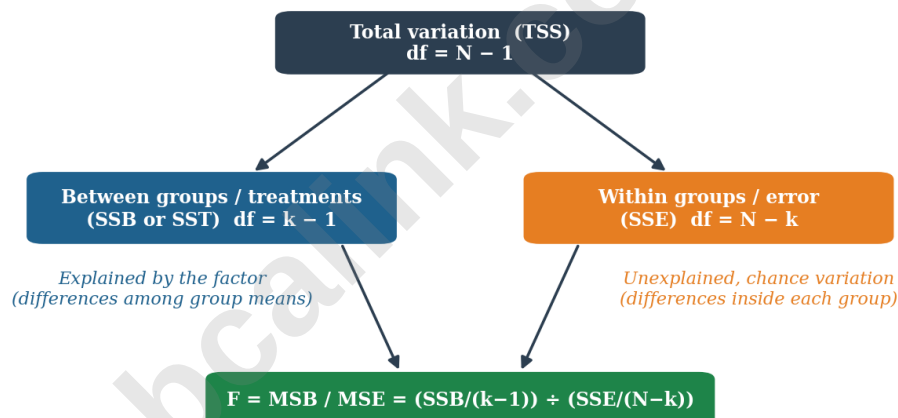


Figure 7.1 — Splitting total variation into between-group and within-group parts

7.1.1 Basic Terminology

Term	Meaning
Experiment	A planned procedure for collecting data under controlled conditions.
Experimental unit	The smallest object to which one treatment is applied (a plot of land, a student, a machine).
Factor	The variable whose effect is studied (fertilizer type, teaching method).
Levels / Treatments	The different values or categories of a factor (fertilizers A, B, C).
Response variable	The measured outcome (yield, marks, time).
Replication	Repeating a treatment on several units to estimate experimental error.
Randomization	Allocating treatments to units by chance to remove bias.
Blocking / local control	Grouping similar units so that known sources of variation can be separated.
Fixed-effect model	The levels of the factor are chosen by the experimenter and conclusions apply <i>only to those levels</i> .

7.1.2 Assumptions of ANOVA

- **Independence** — observations are independent of one another (ensured by randomization).
- **Normality** — the errors (and hence the populations) are normally distributed.
- **Homogeneity of variance** — all populations have the same variance σ^2 .
- **Additivity** (for two-way models without interaction) — the effects of the factors simply add.

Note: ANOVA is fairly robust to mild departures from normality, especially when the groups are of equal size.

7.1.3 Types of Classification

Classification	Factors studied	Model
One-way	One factor with k levels	Completely randomised design (CRD)
Two-way (1 observation per cell)	Two factors, one observation for each combination	No interaction can be estimated
Two-way (m observations per cell)	Two factors, m replicates in each combination	Interaction can be tested

7.2 F-Statistic and its Distribution

The test statistic of ANOVA is the **F-ratio**, named after Fisher. If χ_1^2 and χ_2^2 are *independent* chi-square variables with ν_1 and ν_2 degrees of freedom, then

Definition

$F = (\chi_1^2 / \nu_1) \div (\chi_2^2 / \nu_2)$ follows the F-distribution with (ν_1, ν_2) degrees of freedom.

In ANOVA: $F = MSB / MSE = (\text{mean square between groups}) \div (\text{mean square within groups / error})$.

7.2.1 Properties of the F-Distribution

- It is a **continuous** distribution that takes only **non-negative values** ($0 \leq F < \infty$).
- It is **positively skewed**; the skewness decreases as the degrees of freedom increase.
- Its shape depends on **two parameters**, the numerator d.f. ν_1 and the denominator d.f. ν_2 .
- The mean is $\nu_2 / (\nu_2 - 2)$ for $\nu_2 > 2$ — close to 1 for large ν_2 .
- Reciprocal property: $F_{1-\alpha}(\nu_1, \nu_2) = 1 / F_{\alpha}(\nu_2, \nu_1)$.
- Tables give the **upper-tail critical value** $F_{\alpha}(\nu_1, \nu_2)$, so ANOVA is a **right-tailed test**.

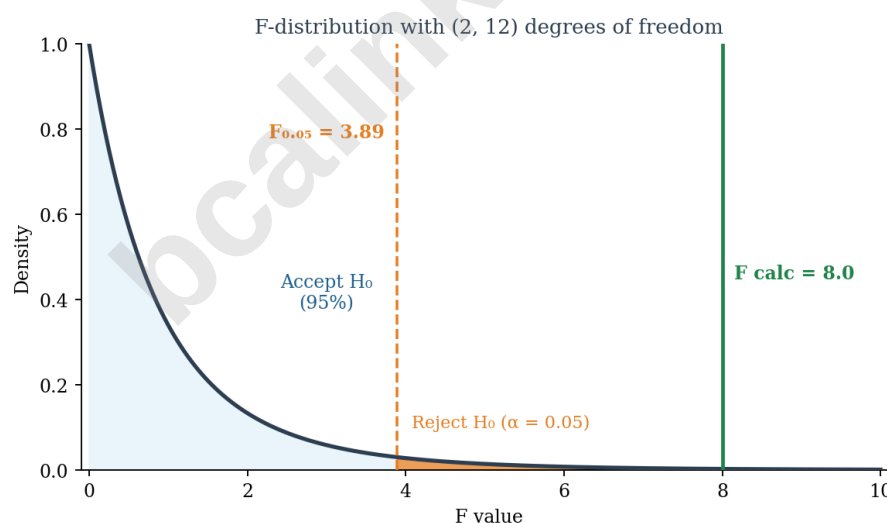


Figure 7.2 — $F(2, 12)$ distribution: calculated $F = 8.0$ falls in the rejection region

7.2.2 Decision Rule

Step	Action
1	State H_0 (all population means equal) and H_1 (at least one mean differs).
2	Choose the level of significance α (usually 0.05).
3	Compute $F = \text{MSB} / \text{MSE}$ from the ANOVA table.
4	Read the critical value $F_{\alpha}(v_1, v_2)$ from the F-table.
5	If $F \text{ calculated} > F \text{ tabulated} \rightarrow$ reject H_0 ; otherwise accept (fail to reject) H_0 .

Note: A large F means the variation between group means is large compared with the natural (error) variation, so the group means are unlikely to be equal.

7.3 Linear Model in ANOVA

Each observation is expressed as the sum of a common mean, the effects of the factors, and a random error. Such an additive representation is a **linear model**.

Design	Linear model	Symbols
One-way	$y_{ij} = \mu + \alpha_i + \varepsilon_{ij}$	$i = 1, \dots, k$ treatments; $j = 1, \dots, n_i$ observations; μ = overall mean; α_i = effect of i -th treatment; ε_{ij} = random error
Two-way, 1 obs per cell	$y_{ij} = \mu + \alpha_i + \beta_j + \varepsilon_{ij}$	α_i = effect of i -th level of factor A (rows); β_j = effect of j -th level of factor B (columns)
Two-way, m obs per cell	$y_{ijk} = \mu + \alpha_i + \beta_j + (\alpha\beta)_{ij} + \varepsilon_{ijk}$	$(\alpha\beta)_{ij}$ = interaction effect of A at level i and B at level j ; $k = 1, \dots, m$ replicates

Assumptions of the fixed-effect model: the effects are fixed constants that sum to zero — $\sum \alpha_i = 0$, $\sum \beta_j = 0$, $\sum_i (\alpha\beta)_{ij} = \sum_j (\alpha\beta)_{ij} = 0$ — and the errors are independent with $\varepsilon_{ij} \sim N(0, \sigma^2)$.

Design	Null hypotheses tested
One-way	$H_0: \alpha_1 = \alpha_2 = \dots = \alpha_k = 0$ (all treatment means are equal)
Two-way	H_0^A : all $\alpha_i = 0$ (factor A has no effect); H_0^B : all $\beta_j = 0$ (factor B has no effect)
Two-way with m obs	The two above, plus H_0^{AB} : all $(\alpha\beta)_{ij} = 0$ (no interaction between A and B)

7.4 One-Way Classification

In a one-way ANOVA the observations are classified by **one factor** only, having k levels (groups). Suppose group i has n_i observations, and $N = n_1 + n_2 + \dots + n_k$. Let T_i be the total of group i and G the grand total.

7.4.1 Computational Formulae

Quantity	Formula
Correction factor (CF)	$CF = G^2 / N$
Total sum of squares	$TSS = \sum \sum y_{ij}^2 - CF$
Between-treatment SS	$SSB = \sum (T_i^2 / n_i) - CF$
Error (within) SS	$SSE = TSS - SSB$
Degrees of freedom	Total: $N - 1$; Between: $k - 1$; Error: $N - k$
Mean squares	$MSB = SSB / (k - 1)$; $MSE = SSE / (N - k)$
Test statistic	$F = MSB / MSE$ with $(k - 1, N - k)$ d.f.

Source of variation	d.f.	Sum of squares	Mean square	F-ratio
Between treatments	$k - 1$	SSB	$MSB = SSB/(k-1)$	$F = MSB/MSE$
Error (within)	$N - k$	SSE	$MSE = SSE/(N-k)$	
Total	$N - 1$	TSS		

Example 7.1 — One-way ANOVA

Yields (kg per plot) of a crop under three fertilizers, five plots each, are given below. Test at the 5% level whether the mean yield is the same for all fertilizers.

Fertilizer	Yields	Total T_i	Mean
A	6, 7, 8, 5, 9	35	7
B	8, 9, 10, 7, 11	45	9
C	5, 6, 4, 7, 3	25	5

Solution

H₀: $\mu_A = \mu_B = \mu_C$ (no difference among fertilizers). **H₁:** at least one mean differs. $\alpha = 0.05$.

Here $k = 3$, $n = 5$ each, $N = 15$. Grand total $G = 35 + 45 + 25 = 105$.

$$CF = 105^2 / 15 = 11025 / 15 = 735.$$

$$\Sigma\Sigma y^2 = (36+49+64+25+81) + (64+81+100+49+121) + (25+36+16+49+9) = 255 + 415 + 135 = 805.$$

$$TSS = 805 - 735 = 70.$$

$$SSB = (35^2 + 45^2 + 25^2)/5 - CF = (1225 + 2025 + 625)/5 - 735 = 775 - 735 = 40.$$

$$SSE = 70 - 40 = 30.$$

Source	d.f.	SS	MS	F calc	F _{0.05} (tab)
Between fertilizers	2	40	20.0	8.00	3.89 (2, 12)
Error	12	30	2.5		
Total	14	70			

Conclusion

Since F calculated (8.00) > F tabulated (3.89), we **reject H₀**. The mean yields of the three fertilizers differ significantly at the 5% level. (Fertilizer B, with mean 9, gave the highest yield.)

Unequal group sizes: the same formulae apply; only the divisor n_i in SSB changes for each group, and N is the sum of all n_i . **Critical difference:** when H_0 is rejected, the pairs of means that differ are found by the least significant difference $LSD = t(\alpha/2, N-k) \times \sqrt{(2 \cdot MSE / n)}$. For Example 7.1, $LSD = 2.179 \times \sqrt{(2 \times 2.5 / 5)} = 2.179$; so $|A - B| = 2$ (not significant), $|B - C| = 4$ and $|A - C| = 2$ — only B versus C exceeds the LSD.

7.5 Two-Way Classification (One Observation per Cell)

Here the data are classified by **two factors**: factor A with r levels (rows) and factor B with c levels (columns). Each row–column combination (cell) has exactly **one observation**, so $N = r \times c$. The total variation splits into three parts — rows, columns and error — and there is *no* way to estimate interaction. Let R_i be the i -th row total, C_j the j -th column total and G the grand total.

Quantity	Formula	d.f.
CF	G^2 / N	—
TSS	$\sum \sum y_{ij}^2 - CF$	$rc - 1$
SSR (rows, factor A)	$\sum R_i^2 / c - CF$	$r - 1$
SSC (columns, factor B)	$\sum C_j^2 / r - CF$	$c - 1$
SSE (error)	$TSS - SSR - SSC$	$(r - 1)(c - 1)$

Source	d.f.	SS	MS	F-ratio
Rows (factor A)	$r - 1$	SSR	$MSR = SSR/(r-1)$	$F_A = MSR / MSE$
Columns (factor B)	$c - 1$	SSC	$MSC = SSC/(c-1)$	$F_B = MSC / MSE$
Error	$(r-1)(c-1)$	SSE	$MSE = SSE/[(r-1)(c-1)]$	
Total	$rc - 1$	TSS		

Example 7.2 — Two-way ANOVA (one observation per cell)

The yield of four wheat varieties (rows) was recorded with three fertilizers (columns). Test at 5% whether (a) varieties and (b) fertilizers differ significantly.

Variety	F_1	F_2	F_3	Row total
V_1	6	7	8	21
V_2	5	6	9	20
V_3	8	10	12	30
V_4	7	9	11	27
Column total	26	32	40	$G = 98$

Solution

H_0^A : variety means are equal. H_0^B : fertilizer means are equal. $\alpha = 0.05$; $r = 4$, $c = 3$, $N = 12$.

$$CF = 98^2 / 12 = 9604 / 12 = 800.33.$$

$$\sum \sum y^2 = 36 + 49 + 64 + 25 + 36 + 81 + 64 + 100 + 144 + 49 + 81 + 121 = 850. \text{ TSS} = 850 - 800.33 = 49.67.$$

$$SSR = (21^2 + 20^2 + 30^2 + 27^2)/4 - CF = 2470/4 - 800.33 = 617.5 - 800.33 = -182.83.$$

$$SSC = (26^2 + 32^2 + 40^2)/3 - CF = 3300/3 - 800.33 = 1100 - 800.33 = 299.67.$$

$$SSE = 49.67 - (-182.83) - 299.67 = 2.00.$$

Source	d.f.	SS	MS	F calc	F _{0.05} (tab)
Varieties (rows)	3	23.00	7.667	23.0	4.76 (3, 6)
Fertilizers (columns)	2	24.67	12.333	37.0	5.14 (2, 6)
Error	6	2.00	0.333		
Total	11	49.67			

Conclusion

(a) $23.0 > 4.76 \rightarrow$ reject H_0^A : the **varieties differ significantly**.

(b) $37.0 > 5.14 \rightarrow$ reject H_0^B : the **fertilizers differ significantly**.

7.6 Two-Way Classification (m Observations per Cell)

When each cell contains **m > 1 replicate observations**, the error can be measured separately from the factor effects, and we can also test the **interaction** between the factors. There are a levels of A, b levels of B and m observations per cell, so $N = abm$. Interaction means that the effect of one factor *depends on the level* of the other factor.

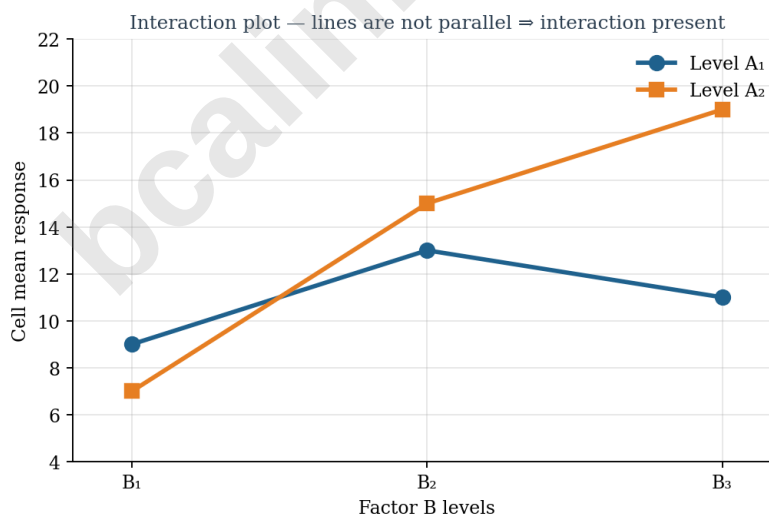


Figure 7.3 — Non-parallel lines indicate an interaction between factors A and B

Let A_i be the total of level i of A (over all j and k), B_j the total of level j of B, T_{ij} the total of cell (i, j) , and G the grand total.

Quantity	Formula	d.f.
CF	G^2 / abm	—
TSS	$\sum \sum \sum y_{ijk}^2 - CF$	$abm - 1$
SSA (factor A)	$\sum A_i^2 / (bm) - CF$	$a - 1$
SSB (factor B)	$\sum B_j^2 / (am) - CF$	$b - 1$
SS between cells (subtotals)	$\sum \sum T_{ij}^2 / m - CF$	$ab - 1$
SSAB (interaction)	$SS \text{ cells} - SSA - SSB$	$(a - 1)(b - 1)$
SSE (error)	$TSS - SS \text{ cells}$	$ab(m - 1)$

Source	d.f.	SS	MS	F-ratio
Factor A	$a - 1$	SSA	$MSA = SSA / (a - 1)$	$F_A = MSA / MSE$
Factor B	$b - 1$	SSB	$MSB = SSB / (b - 1)$	$F_B = MSB / MSE$
Interaction A $\times$ B	$(a - 1)(b - 1)$	SSAB	$MSAB = SSAB / [(a - 1)(b - 1)]$	$F_{AB} = MSAB / MSE$
Error	$ab(m - 1)$	SSE	$MSE = SSE / [ab(m - 1)]$	
Total	$abm - 1$	TSS		

*Note: Test the **interaction first**. If it is significant, the main effects cannot be interpreted separately, because the effect of A changes with the level of B.*

Example 7.3 — Two-way ANOVA with two observations per cell

A study uses two machines A_1, A_2 and three operators B_1, B_2, B_3 , with two output readings per combination. Analyse the data at 5%.

	B_1	B_2	B_3	Total A_i
A_1	8, 10 (18)	12, 14 (26)	10, 12 (22)	66
A_2	6, 8 (14)	14, 16 (30)	18, 20 (38)	82
Total B_j	32	56	60	G = 148

Solution

Here $a = 2$, $b = 3$, $m = 2$, $N = 12$. Cell totals are shown in brackets.

$$CF = 148^2 / 12 = 21904 / 12 = 1825.33.$$

$$\Sigma\Sigma y^2 = (64+100+144+196+100+144) + (36+64+196+256+324+400) = 748 + 1276 = 2024.$$

$$TSS = 2024 - 1825.33 = 198.67.$$

$$SSA = (66^2 + 82^2)/6 - CF = (4356 + 6724)/6 - 1825.33 = 1846.67 - 1825.33 = 21.33.$$

$$SSB = (32^2 + 56^2 + 60^2)/4 - CF = (1024 + 3136 + 3600)/4 - 1825.33 = 1940 - 1825.33 = 114.67.$$

$$SS \text{ cells} = (18^2 + 26^2 + 22^2 + 14^2 + 30^2 + 38^2)/2 - CF = (324+676+484+196+900+1444)/2 - 1825.33 = 2012 - 1825.33 = 186.67.$$

$$SSAB = 186.67 - 21.33 - 114.67 = 50.67. SSE = 198.67 - 186.67 = 12.00.$$

Source	d.f.	SS	MS	F calc	F _{0.05} (tab)
Machines (A)	1	21.33	21.33	10.67	5.99 (1, 6)
Operators (B)	2	114.67	57.33	28.67	5.14 (2, 6)
Interaction A × B	2	50.67	25.33	12.67	5.14 (2, 6)
Error	6	12.00	2.00		
Total	11	198.67			

Conclusion

All three calculated F values exceed their critical values, so the machines, the operators and the **interaction** are all significant at the 5% level. Because the interaction is significant (Figure 7.3: the lines are not parallel), the best operator depends on the machine used — the effect of operator B₃ is much larger on machine A₂ than on A₁.

7.6.1 Comparison of the Three Designs

Feature	One-way	Two-way (1 obs)	Two-way (m obs)
Factors	1	2	2
Model	$\mu + \alpha_i + \varepsilon_{ij}$	$\mu + \alpha_i + \beta_j + \varepsilon_{ij}$	$\mu + \alpha_i + \beta_j + (\alpha\beta)_{ij} + \varepsilon_{ijk}$
Error d.f.	$N - k$	$(r-1)(c-1)$	$ab(m-1)$
F-tests	1	2	3 (A, B, A×B)
Interaction tested?	Not applicable	No	Yes
Extra observations needed	No	No	Yes (replicates)

Key Takeaways — Unit 7

- ✓ ANOVA tests equality of three or more means by splitting total variation into between-group and error parts.
- ✓ $F = \text{MSB} / \text{MSE}$ follows $F(v_1, v_2)$; it is right-tailed, non-negative and positively skewed. Reject H_0 if $F_{\text{calc}} > F_{\text{tab}}$.
- ✓ Linear models (fixed effect): $y_{ij} = \mu + \alpha_i + \varepsilon_{ij}$; $y_{ij} = \mu + \alpha_i + \beta_j + \varepsilon_{ij}$; $y_{ijk} = \mu + \alpha_i + \beta_j + (\alpha\beta)_{ij} + \varepsilon_{ijk}$.
- ✓ Steps: $CF = G^2/N \rightarrow \text{TSS} \rightarrow \text{treatment SS} \rightarrow \text{SSE by subtraction} \rightarrow \text{mean squares} \rightarrow F \rightarrow$ compare with table.
- ✓ One-way d.f.: $k-1, N-k$. Two-way (1 obs): $r-1, c-1, (r-1)(c-1)$. Two-way (m obs): $a-1, b-1, (a-1)(b-1), ab(m-1)$.
- ✓ Only with $m > 1$ observations per cell can interaction be tested; check interaction before interpreting main effects.

— End of Unit 7 —