

UNIT 3

Sequence and Series

(7 Hours) — Mathematics I (CACS104), Year/Sem I/I

This unit introduces sequences and series — Arithmetic, Geometric, and Harmonic — along with the properties that connect them. We study the Arithmetic Mean (A.M.), Geometric Mean (G.M.), and Harmonic Mean (H.M.), the relations among them, and the sum of an infinite geometric series. The unit closes with Taylor's theorem and Taylor's series (stated without proof) and the exponential series, which together form the bridge between algebra and calculus-based approximation of functions.

1. Sequences — Basic Ideas

A sequence is an ordered list of numbers $a_1, a_2, a_3, \dots, a_n, \dots$ arranged according to some definite rule, where each number is called a term of the sequence. A sequence having a finite number of terms is a finite sequence; one that continues indefinitely is an infinite sequence.

When the terms of a sequence are added together with plus signs, the resulting expression $a_1 + a_2 + a_3 + \dots + a_n + \dots$ is called a series. If the sequence is finite the series is a finite series; if the sequence is infinite the series is an infinite series.

Note: The general (nth) term of a sequence is usually written as a_n , and it expresses each term as a function of its position n .

2. Arithmetic Sequence and Series

An Arithmetic Sequence (A.P.) is a sequence in which every term after the first is obtained by adding a fixed number, called the common difference (d), to the preceding term. If the first term is a and the common difference is d , the sequence is written as $a, a + d, a + 2d, a + 3d, \dots$

General (nth) term:

$$a_n = a + (n - 1)d$$

The corresponding Arithmetic Series is $a + (a+d) + (a+2d) + \dots + [a+(n-1)d]$, and its sum of the first n terms is given by:

$$S_n = n/2 [2a + (n - 1)d] = n/2 (a + l)$$

where $l = a + (n - 1)d$ is the last (nth) term.

Example 1

Find the 10th term and the sum of the first 10 terms of the A.P.: 3, 7, 11, 15, ...

Solution

Here $a = 3$, $d = 4$, $n = 10$.

$$a_{10} = a + 9d = 3 + 9(4) = 3 + 36 = 39$$

$$S_{10} = n/2 [2a + (n-1)d] = 10/2 [2(3) + 9(4)] = 5[6 + 36] = 5 \times 42 = 210$$

Properties of Arithmetic Sequences:

- If a constant is added to, or subtracted from, every term of an A.P., the resulting sequence is still an A.P. with the same common difference.
- If every term of an A.P. is multiplied (or divided) by a nonzero constant k , the resulting sequence is an A.P. with common difference kd .
- Three numbers x , y , z are in A.P. if and only if $2y = x + z$, i.e. y is the arithmetic mean of x and z .
- The sum of terms equidistant from the beginning and end of a finite A.P. is always the same, and equals (first term + last term).

3. Geometric Sequence and Series

A Geometric Sequence (G.P.) is a sequence in which every term after the first is obtained by multiplying the preceding term by a fixed nonzero number called the common ratio (r). If the first term is a and common ratio r , the sequence is a , ar , ar^2 , ar^3 , ...

General (n th) term:

$$a_n = a \cdot r^{n-1}$$

The Geometric Series $a + ar + ar^2 + \dots + ar^{n-1}$ has sum of first n terms:

$$S_n = a(r^n - 1)/(r - 1), \text{ for } r \neq 1 \text{ (equivalently, } a(1 - r^n)/(1 - r) \text{)}$$

If $r = 1$, then $S_n = na$, since every term equals a .

Example 2

Find the 6th term and the sum of the first 6 terms of the G.P.: 2, 6, 18, 54, ...

Solution

Here $a = 2$, $r = 3$, $n = 6$.

$$a_6 = a \cdot r^5 = 2 \times 3^5 = 2 \times 243 = 486$$

$$S_6 = a(r^n - 1)/(r - 1) = 2(3^6 - 1)/(3 - 1) = 2(729 - 1)/2 = 728$$

Sum of an Infinite Geometric Series:

When $|r| < 1$, the terms of a G.P. shrink towards zero as $n \rightarrow \infty$, and the partial sum S_n approaches a finite limit. The sum to infinity is then:

$$S_\infty = a / (1 - r), \text{ valid only when } -1 < r < 1$$

If $|r| \geq 1$ (and $r \neq \dots$ trivial cases), the series does not converge and the sum to infinity does not exist.

Example 3

Find the sum to infinity of $8 + 4 + 2 + 1 + \dots$.

Solution

Here $a = 8$, $r = 1/2$, and $|r| < 1$, so the series converges.

$$S_\infty = a/(1-r) = 8/(1 - 1/2) = 8/(1/2) = 16$$

Properties of Geometric Sequences:

- If every term of a G.P. is multiplied (or divided) by a nonzero constant, the resulting sequence remains a G.P. with the same common ratio.
- The reciprocals of the terms of a G.P. also form a G.P., with common ratio $1/r$.
- Three numbers x, y, z are in G.P. if and only if $y^2 = xz$, i.e. y is the geometric mean of x and z .
- If each term of a G.P. is raised to the same power k , the resulting sequence is also a G.P. with common ratio r^k .

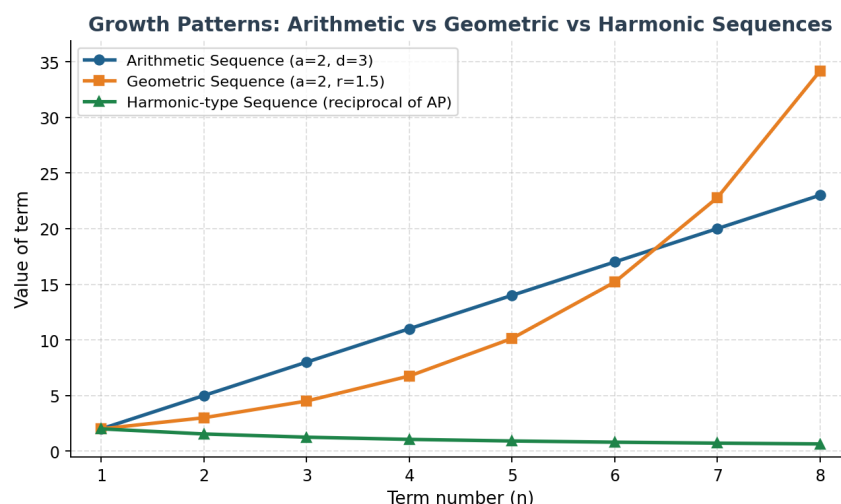


Figure 1: Growth patterns of Arithmetic, Geometric, and Harmonic-type sequences compared term by term.

4. Harmonic Sequence and Series

A Harmonic Sequence (H.P.) is a sequence whose reciprocals form an Arithmetic Sequence. That is, $a_1, a_2, a_3, \dots$ is an H.P. if $1/a_1, 1/a_2, 1/a_3, \dots$ is an A.P.

If the corresponding A.P. of reciprocals has first term $1/a$ and common difference d , then the n th term of the H.P. is:

$$a_n = 1 / [1/a + (n - 1)d]$$

There is no simple closed-form formula for the sum of a Harmonic Series analogous to A.P. or G.P.; sums of Harmonic Series are usually evaluated by taking reciprocals, converting to the associated A.P., and working from there.

Example 4

Show that $1/2, 1/5, 1/8, 1/11, \dots$ is a Harmonic Sequence, and find its 5th term.

Solution

The reciprocals are 2, 5, 8, 11, ... which form an A.P. with $a = 2, d = 3$.

So the given sequence is indeed an H.P.

5th term of reciprocal A.P.: $a_5 = 2 + 4(3) = 14$, so the 5th term of the H.P. is $1/14$.

Note: A Harmonic Mean question is best solved by switching to reciprocals, applying A.P. rules, then inverting the result back at the end.

5. Arithmetic, Geometric and Harmonic Means

For two positive numbers a and b , the three classical means are defined as:

Mean	Formula (for two numbers a, b)	Meaning
Arithmetic Mean (A.M.)	$A = (a + b) / 2$	Average of the two numbers
Geometric Mean (G.M.)	$G = \sqrt{(ab)}$	Middle term of a G.P. between a, b
Harmonic Mean (H.M.)	$H = 2ab / (a + b)$	Reciprocal of A.M. of reciprocals

For n positive numbers $a_1, a_2, \dots, a_n$, these generalize to:

$$A = (a_1 + a_2 + \dots + a_n)/n, G = (a_1 \cdot a_2 \cdot \dots \cdot a_n)^{(1/n)}, H = n / (1/a_1 + 1/a_2 + \dots + 1/a_n)$$

Relations among A.M., G.M. and H.M.:

- $A \geq G \geq H$, with equality holding if and only if all the numbers are equal.
- $G^2 = A \cdot H$ (the Geometric Mean is itself the Geometric Mean of A and H, for two numbers).
- For two positive numbers a, b: $A \cdot H = ab = G^2$.

Example 5

Find the A.M., G.M. and H.M. of 4 and 16, and verify that $A \geq G \geq H$ and $G^2 = A \cdot H$.

Solution

$$A = (4+16)/2 = 10$$

$$G = \sqrt{(4 \times 16)} = \sqrt{64} = 8$$

$$H = 2(4)(16)/(4+16) = 128/20 = 6.4$$

$$\text{Check: } A \geq G \geq H \Rightarrow 10 \geq 8 \geq 6.4 \checkmark$$

$$\text{Check: } G^2 = 64, A \cdot H = 10 \times 6.4 = 64 \Rightarrow G^2 = A \cdot H \checkmark$$

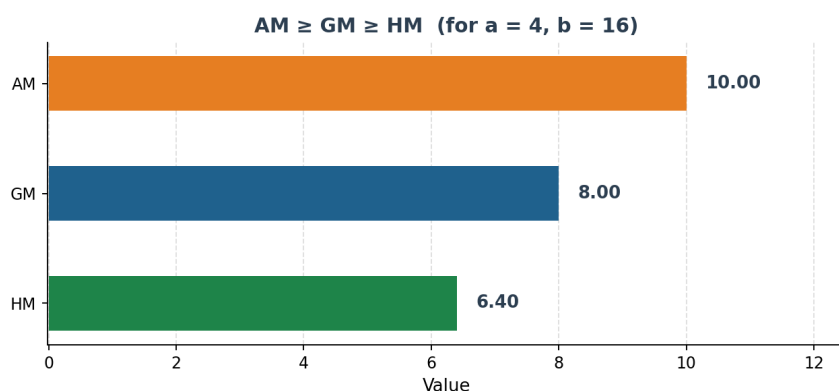


Figure 2: Visual comparison confirming $A.M. \geq G.M. \geq H.M.$ for $a = 4, b = 16$.

Inserting means: To insert n Arithmetic Means between a and b, treat a, $A_1, A_2, \dots, A_n, b$ as an A.P. of (n+2) terms with common difference $d = (b-a)/(n+1)$. To insert n Geometric Means

between a and b , treat $a, G_1, G_2, \dots, G_n, b$ as a G.P. of $(n+2)$ terms with common ratio $r = (b/a)^{1/(n+1)}$.

6. Taylor's Theorem and Taylor's Series

Taylor's Theorem (statement, without proof): If a function $f(x)$ is continuous and possesses continuous derivatives up to order $(n+1)$ in an interval containing the point $x = a$, then for any x in that interval:

$$f(x) = f(a) + (x-a)f'(a) + (x-a)^2/2! \cdot f''(a) + (x-a)^3/3! \cdot f'''(a) + \dots + (x-a)^n/n! \cdot f^{(n)}(a) + R_n$$

where R_n is the remainder term after n terms. When $R_n \rightarrow 0$ as $n \rightarrow \infty$, the expansion becomes an infinite series called Taylor's Series of $f(x)$ about the point $x = a$:

$$f(x) = \sum_{k=0}^{\infty} [(x-a)^k / k!] \cdot f^{(k)}(a)$$

This series expresses the value of a function at any nearby point x purely in terms of the function's value and derivatives at the single reference point a — making it a fundamental tool for approximating functions using polynomials.

Note: When the reference point is taken as $a = 0$, Taylor's Series is called a Maclaurin's Series: $f(x) = f(0) + x f'(0) + x^2/2! f''(0) + \dots$.

Example 6

Write the first four terms of the Taylor's Series of $f(x) = \ln(x)$ about $x = 1$.

Solution

$$f(x) = \ln x, f(1) = 0$$

$$f'(x) = 1/x, f'(1) = 1$$

$$f''(x) = -1/x^2, f''(1) = -1$$

$$f'''(x) = 2/x^3, f'''(1) = 2$$

$$\text{So: } \ln x = 0 + (x-1)(1) + (x-1)^2/2!(-1) + (x-1)^3/3!(2) + \dots$$

$$= (x-1) - (x-1)^2/2 + (x-1)^3/3 - \dots$$

7. Exponential Series

The Exponential Series is obtained by taking the Maclaurin's (Taylor's, about $a = 0$) expansion of $f(x) = e^x$. Since every derivative of e^x is e^x itself, and $e^x = 1$ at $x = 0$, all derivatives at $x = 0$ equal 1. Substituting into the Taylor/Maclaurin expansion gives:

$$e^x = 1 + x + x^2/2! + x^3/3! + x^4/4! + \dots + x^n/n! + \dots \text{ (valid for all real } x \text{)}$$

Putting $x = 1$ gives the important special value:

$$e = 1 + 1 + 1/2! + 1/3! + 1/4! + \dots = 2.71828\dots$$

Example 7

Use the Exponential Series to estimate $e^{0.5}$ correct to 4 decimal places using terms up to $x^4/4!$.

Solution

$$e^{0.5} \approx 1 + 0.5 + 0.5^2/2! + 0.5^3/3! + 0.5^4/4!$$

$$= 1 + 0.5 + 0.125 + 0.0208 + 0.0026$$

$$\approx 1.6484 \text{ (actual value } e^{0.5} \approx 1.6487, \text{ so the approximation is already very close)}$$

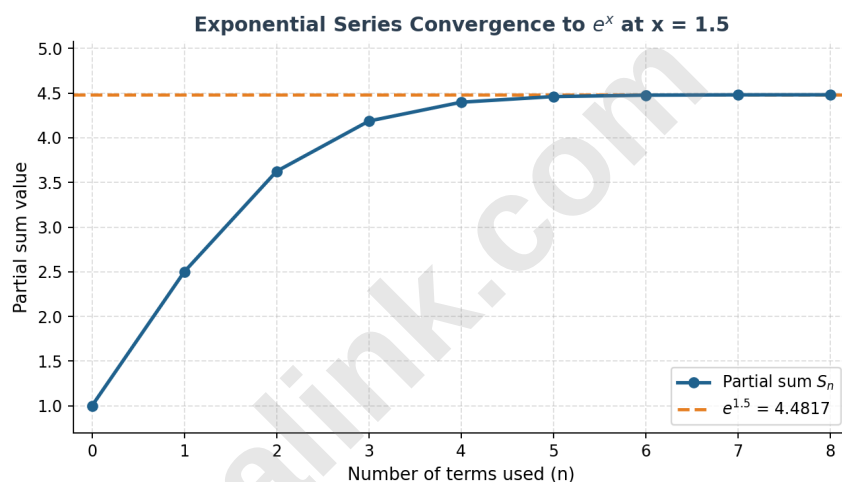


Figure 3: Partial sums of the Exponential Series converging rapidly towards $e^{1.5}$ as more terms are added.

Related standard series (each obtained similarly from Taylor/Maclaurin expansion, and worth remembering alongside the exponential series):

- $\sin x = x - x^3/3! + x^5/5! - x^7/7! + \dots$
- $\cos x = 1 - x^2/2! + x^4/4! - x^6/6! + \dots$
- $\ln(1 + x) = x - x^2/2 + x^3/3 - x^4/4 + \dots$ (for $-1 < x \leq 1$)

Key Takeaways

- ✓ A.P.: $a_n = a + (n-1)d$, $S_n = n/2[2a + (n-1)d]$
- ✓ G.P.: $a_n = a \cdot r^{n-1}$, $S_n = a(r^n - 1)/(r - 1)$, $S_\infty = a/(1-r)$ for $|r| < 1$
- ✓ H.P.: a sequence whose reciprocals form an A.P. — solve via reciprocal A.P.
- ✓ For two positive numbers: $A \geq G \geq H$ always, and $G^2 = A \cdot H$
- ✓ Taylor's Series expands $f(x)$ about $x = a$ using $f(a)$ and its derivatives at a
- ✓ Exponential Series: $e^x = 1 + x + x^2/2! + x^3/3! + \dots$, valid for all real x

— *End of Unit 3* —

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