

UNIT 7

Game Theory

(7 Hours) — Operations Research

Whenever two competitors make decisions that affect each other — two companies setting prices, two political parties choosing campaign strategies, two army commanders planning moves, or even two players in a simple coin-matching game — each side's best choice depends on what the other side is expected to do. Game Theory is the branch of Operations Research that provides a mathematical framework for analysing such situations of conflict and competition, where the outcome for each participant depends not only on their own decision but also on the decisions of others. The word 'game' here does not mean entertainment — it refers to any competitive situation with well-defined rules and outcomes.

7.1 Introduction to Game Theory

Game Theory was formally developed by John von Neumann and Oskar Morgenstern in their 1944 book *Theory of Games and Economic Behavior*. It studies situations where two or more rational decision-makers ('players') compete, each trying to maximise their own gain (or minimise their own loss), fully aware that the other player is doing the same. Unlike ordinary decision-making under uncertainty (where 'nature' is unpredictable but not deliberately hostile), in game theory the opponent is an intelligent, purposeful adversary who will choose their best response to whatever we do.

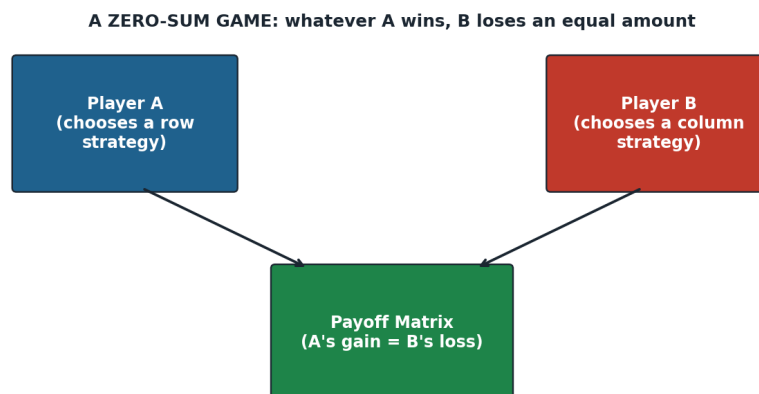


Fig. 7.1 — The basic structure of a two-person game: each player independently chooses a strategy, and the combination of choices determines the payoff, which in a zero-sum game is a direct transfer from one player to the other.

7.1.1 Basic Terminology

Term	Meaning
Player	A participant in the game (an individual, firm, or country) who makes decisions
Strategy	A complete plan of action a player may follow; choosing one course of action out of all those available
Pure Strategy	A decision rule in which a player always selects one particular strategy, with certainty
Mixed Strategy	A decision rule in which a player selects among several strategies according to fixed probabilities, rather than choosing just one strategy every time
Payoff	The outcome (usually expressed in money or points) resulting from a particular combination of strategies chosen by the players
Payoff Matrix	A table showing the payoff to a player for every possible combination of strategies chosen by both players
Value of the Game	The expected payoff to a player when both players use their optimal strategies
Optimal Strategy	The course of action (pure or mixed) that puts a player in the most preferred position, regardless of what the opponent does

7.2 Characteristics of Game Theory

A competitive situation qualifies as a 'game' in the formal sense studied here when it satisfies the following characteristics:

- There are a finite (or at least well-defined) number of competitors, called players — this unit focuses on exactly two players.
- Each player has a finite list of possible courses of action (strategies) available to choose from.
- A play of the game occurs when each player chooses one of their available strategies; the choices need not be known to the other player in advance.
- The outcome (payoff) of the game is determined by the particular combination of strategies chosen by all the players, and is known in advance for every possible combination.
- The players act rationally and intelligently, each one trying to maximise their own gain or minimise their own loss.
- The game is played repeatedly under identical conditions (conceptually), which justifies looking for a strategy that is best 'on average' or 'in the long run'.

Note: Game Theory does not predict who will actually win a single play — it identifies the strategy that is rational and safest for a player to adopt, assuming the opponent is equally rational and will exploit any weakness.

7.2.1 Classification of Games

Basis of Classification	Categories
Number of players	Two-person game; n-person game (this unit covers two-person games only)
Sum of payoffs	Zero-sum game (one player's gain exactly equals the other's loss); Non-zero-sum game (total payoff can vary)
Number of strategies	Finite game (limited strategies, e.g. an $m \times n$ payoff matrix); Infinite game (unlimited choices, e.g. a continuous range of prices)

7.3 Two-Person Zero-Sum Games

This unit focuses entirely on the two-person zero-sum game, the simplest and most widely studied class of competitive games. 'Two-person' means exactly two players, conventionally called A and B. 'Zero-sum' means that whatever one player wins, the other player loses an exactly equal amount — the two payoffs always add up to zero. This means we only need one payoff matrix (conventionally written from Player A's point of view: a positive entry means A wins that amount from B, and B's payoff is simply the negative of the same number).

If A has m strategies ($A_1, A_2, \dots, A_m$) and B has n strategies ($B_1, B_2, \dots, B_n$), the game can be represented by an $m \times n$ payoff matrix, where entry a_{ij} (written here simply as the value in row i , column j) represents the payoff to A when A plays strategy A_i and B plays strategy B_j .

7.3.1 The Minimax (or Maximin-Minimax) Criterion

Since A wants to maximise their own payoff while B wants to minimise A's payoff (equivalently, maximise their own gain, since it's zero-sum), both players adopt a cautious 'best worst-case' approach:

- Player A's approach (Maximin): For each of A's strategies (each row), find the minimum possible payoff (the worst case, assuming B plays optimally against A). Then A chooses the strategy (row) that gives the maximum of these row-minimums. This guaranteed value is called A's maximin value.
- Player B's approach (Minimax): For each of B's strategies (each column), find the maximum possible payoff to A (the worst case for B). Then B chooses the strategy (column) that gives the minimum of these column-maximums. This guaranteed value is called B's minimax value.

It can be proven that maximin value $\leq$ minimax value always. When the two values are equal, this common value is called the value of the game, and a saddle point exists — meaning both players have a pure (single, fixed) optimal strategy, and neither player can do better by

deviating, so the game is said to be strictly determined.

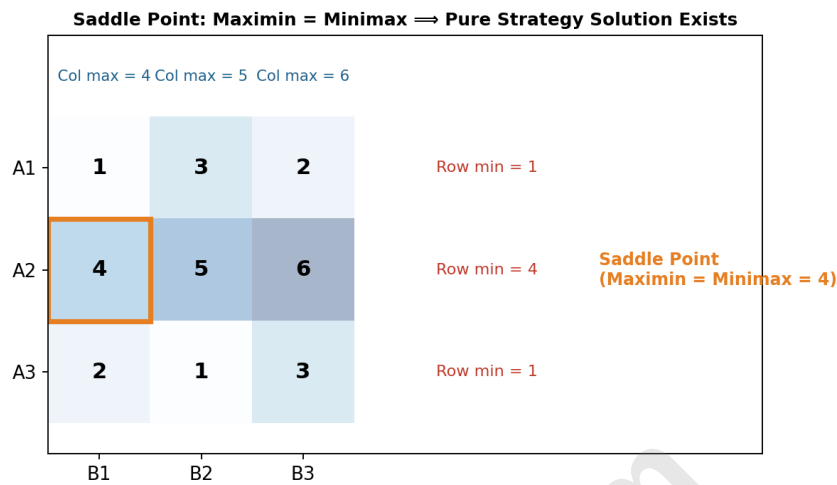


Fig. 7.2 — A saddle point is the cell that is simultaneously the minimum of its row and the maximum of its column; here $\text{Maximin} = \text{Minimax} = 4$, so both players' optimal pure strategies are A2 and B1, and the value of the game is 4.

7.4 Pure Strategy — Games with a Saddle Point

A game has a pure-strategy solution whenever a saddle point exists. The solution procedure is simple and entirely arithmetic:

- Step 1: Write down the row minimums for each row of the payoff matrix, and identify the maximin (the largest of these row minimums).
- Step 2: Write down the column maximums for each column, and identify the minimax (the smallest of these column maximums).
- Step 3: If $\text{maximin} = \text{minimax}$, a saddle point exists at that cell. This common value is the value of the game (V).
- Step 4: The optimal strategy for A is the row containing the saddle point; the optimal strategy for B is the column containing the saddle point.

Example 7.1 — Solving a Game with a Saddle Point (Pure Strategy)

Solve the following game whose payoff is given to Player A:

	B1	B2	B3	Row Min
A1	1	3	2	1

	B1	B2	B3	Row Min
A2	4	5	6	4
A3	2	1	3	1
Col Max	4	5	6	

Solution:

Row minimums: $A1 = 1$, $A2 = 4$, $A3 = 1$. Maximin = $\max(1, 4, 1) = 4$ (occurs at row A2).

Column maximums: $B1 = 4$, $B2 = 5$, $B3 = 6$. Minimax = $\min(4, 5, 6) = 4$ (occurs at column B1).

Since Maximin = Minimax = 4, a saddle point exists at the cell (A2, B1).

Conclusion: Optimal strategy for A is A2; optimal strategy for B is B1; Value of the game $V = 4$ (A is guaranteed to win at least 4 units per play if both play optimally).

Note: If the game has more than one saddle point, all such points will give the same value of the game; A and B simply have more than one equally good optimal strategy to choose from.

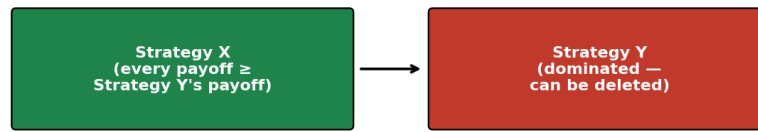
7.5 Dominance Theory (Rule of Dominance)

Many real payoff matrices are larger than 2×2 or 3×3 , which makes them tedious to solve directly. The Principle (Rule) of Dominance is a simplification technique used to reduce the size of a payoff matrix before solving it, by eliminating strategies that a rational player would never use.

7.5.1 Dominance Rules

- For Player A (the maximiser, rows): If every entry in row p is $\geq$ the corresponding entry in row q, then row p dominates row q, and row q can be deleted (A would never choose the worse row, since p is at least as good in every situation).
- For Player B (the minimiser, columns): If every entry in column r is $\leq$ the corresponding entry in column s, then column r dominates column s, and column s can be deleted (B would never choose the worse column, since r always costs B less or the same).
- A row or column can also be dominated by a suitable average or convex combination of two or more other rows/columns, even if no single row/column dominates it outright. This is a more advanced extension of the same idea.

Dominance Principle



Rule: For Player A (maximiser), a row that is always $\geq$ another row dominates it. For Player B (minimiser), a column that is always $\leq$ another column dominates it. Dominated rows/columns can be removed without changing the game's solution.

REMOVED

Fig. 7.3 — If every payoff in one row is at least as good as the corresponding payoff in another row, the weaker row is dominated and can be safely deleted, shrinking the payoff matrix without changing its solution.

Example 7.2 — Reducing a Game Using Dominance

Reduce and solve the following game (payoff to A):

	B1	B2	B3	B4
A1	3	2	4	0
A2	3	4	2	4
A3	4	2	4	0

Solution:

Compare rows: $A3 = (4, 2, 4, 0)$ and $A1 = (3, 2, 4, 0)$. Every entry of A3 is $\geq$ the corresponding entry of A1 ($4 \geq 3, 2 = 2, 4 = 4, 0 = 0$). So A3 dominates A1; delete row A1.

Remaining matrix has rows $A2 = (3, 4, 2, 4)$ and $A3 = (4, 2, 4, 0)$. Now compare columns for B (B wants smaller values). Column $B3 = (2, 4)$ and column $B1 = (3, 4)$: $B3 \leq B1$ in both rows ($2 \leq 3, 4 = 4$). So B3 dominates B1 for player B; delete column B1.

Remaining matrix (rows A2, A3; columns B2, B3, B4): $A2 = (4, 2, 4)$, $A3 = (2, 4, 0)$. Column $B4 = (4, 0)$ compared with column $B3 = (2, 4)$: neither dominates outright (B4 is better in row A3 but worse in row A2), so no further simple dominance applies; the remaining 2×3 (or further-reduced) matrix can now be solved by the methods of Sections 7.6–7.7 below — dominance has still substantially shrunk the problem from 3×4 down to a smaller matrix.

Note: Always apply dominance before attempting mixed-strategy calculations — a smaller matrix is always easier (and less error-prone) to solve.

7.6 Mixed Strategies — Games Without a Saddle Point

When $\text{Maximin} \neq \text{Minimax}$, no saddle point exists, and neither player has a single best pure strategy — whichever pure strategy a player commits to, the opponent can exploit it. The solution is for each player to play several strategies at random, according to carefully chosen fixed probabilities, called a mixed strategy, so that the opponent can never predict the next move and cannot gain an advantage by guessing.

7.6.1 Algebraic Method for 2×2 Games

For a 2×2 game with no saddle point, written generally as:

	B1	B2
A1	a	b
A2	c	d

Let p = probability A plays A1 (so $1-p$ = probability A plays A2), and let q = probability B plays B1 (so $1-q$ = probability B plays B2). The optimal mixed-strategy probabilities and the value of the game are given directly by the formulas:

$$p = (d - c) / (a + d - b - c) \text{ and } 1-p = (a - b) / (a + d - b - c)$$

$$q = (d - b) / (a + d - b - c) \text{ and } 1-q = (a - c) / (a + d - b - c)$$

$$\text{Value of the game, } V = (ad - bc) / (a + d - b - c)$$

Algebraic Method for Solving a 2×2 Mixed-Strategy Game

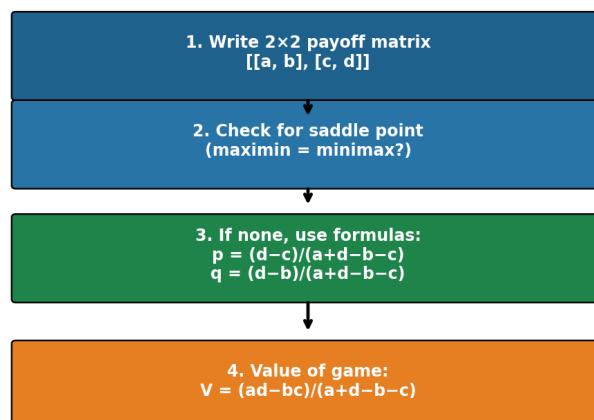


Fig. 7.4 — The four-step algebraic method for solving any 2×2 mixed-strategy game once it is confirmed that no saddle point exists.

Example 7.3 — Solving a 2×2 Game by the Algebraic Method

Solve the following game (payoff to A):

	B1	B2
A1	6	2
A2	3	5

Solution:

First check for a saddle point: row minimums are $A1=2$, $A2=3$, so $\text{Maximin} = 3$. Column maximums are $B1=6$, $B2=5$, so $\text{Minimax} = 5$. Since $\text{Maximin} (3) \neq \text{Minimax} (5)$, no saddle point exists — we must use mixed strategies.

Here $a=6$, $b=2$, $c=3$, $d=5$. Denominator = $a+d-b-c = 6+5-2-3 = 6$.

$p = (d-c)/6 = (5-3)/6 = 2/6 = 1/3$. So A should play A1 with probability $1/3$ and A2 with probability $2/3$.

$q = (d-b)/6 = (5-2)/6 = 3/6 = 1/2$. So B should play B1 with probability $1/2$ and B2 with probability $1/2$.

$V = (ad-bc)/6 = (6 \times 5 - 2 \times 3)/6 = (30-6)/6 = 24/6 = 4$.

Conclusion: A's optimal mixed strategy is $(1/3, 2/3)$; B's optimal mixed strategy is $(1/2, 1/2)$; the value of the game is 4, meaning A wins an average of 4 units per play if both follow these probabilities over many repetitions.

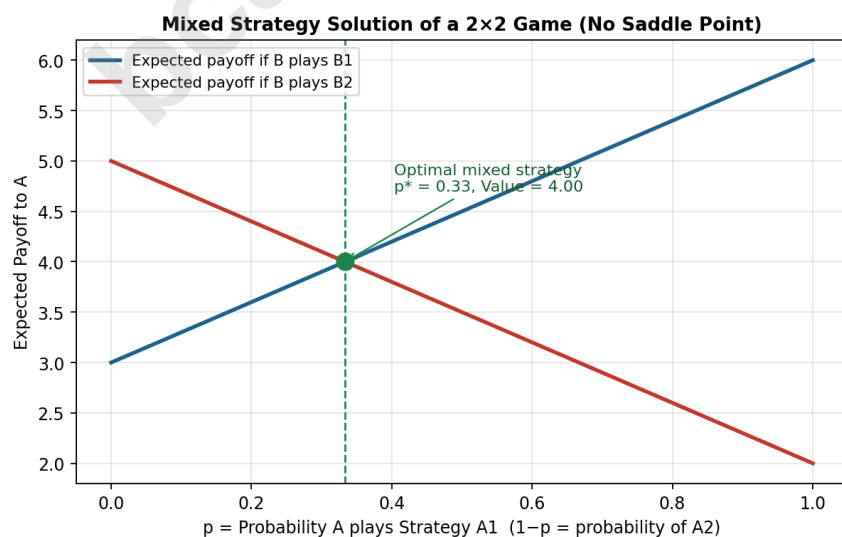


Fig. 7.5 — Graphical confirmation of Example 7.3: the two expected-payoff lines for B's strategies intersect exactly at A's optimal mixing probability, giving the same value of the game found algebraically.

7.6.2 Graphical Method for 2×n (or m×2) Games

When one player has exactly 2 strategies and the other has n (more than 2) strategies, dominance may not reduce the game all the way down to 2×2 . The graphical method handles this directly.

7.6.3 Steps of the Graphical Method (for a $2 \times n$ game)

- Step 1: Let p = probability that A plays strategy A1 (so $1-p$ = probability A plays A2). On a graph, draw two parallel vertical axes at $p = 0$ and $p = 1$.
- Step 2: For each of B's n strategies, plot the expected payoff to A as a straight line in terms of p — this line simply joins the payoff value at $p=0$ to the payoff value at $p=1$ for that particular column.
- Step 3: Since B chooses whichever column is worst for A, A's guaranteed payoff at any value of p is the lowest of all these lines at that point — this traces out the lower envelope of the n lines.
- Step 4: A wants to choose p to make this lower envelope as high as possible — so find the highest point on the lower envelope. This point gives A's optimum mixing probability p^* and the value of the game.
- Step 5: The two (at most) lines that cross exactly at this highest point identify the only two strategies of B that B needs to actually consider — every other strategy of B is effectively dominated and can be dropped. The game then reduces to an ordinary 2×2 game between A's two strategies and these two surviving strategies of B, which can be solved algebraically (Section 7.6.1) to get exact, clean fractions.

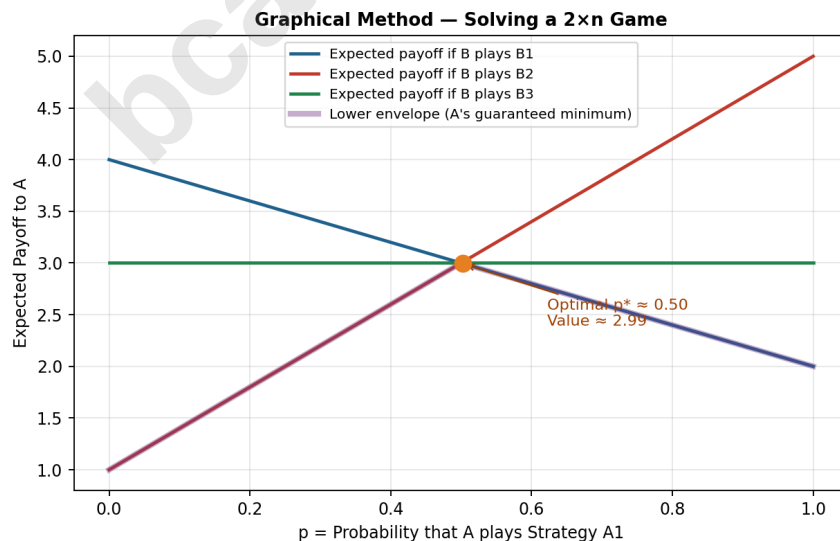


Fig. 7.6 — In a 2×3 game, three straight lines represent B's three strategies; the lower envelope (A's guaranteed minimum payoff) is highest where two of these lines cross, identifying both A's optimal p^* and the two relevant strategies of B.

Example 7.4 — Solving a 2×3 Game by the Graphical Method

Player A has 2 strategies and Player B has 3 strategies, with the following payoff matrix (payoff to A). Find the value of the game using the graphical method.

	B1	B2	B3
A1	2	5	3
A2	4	1	3

Solution:

Let p = probability A plays A1. Expected payoff to A for each column of B, as a function of p , runs from the A1-row value (at $p=1$) to the A2-row value (at $p=0$):

If B plays B1: payoff = $2p + 4(1-p) = 4 - 2p$.

If B plays B2: payoff = $5p + 1(1-p) = 1 + 4p$.

If B plays B3: payoff = $3p + 3(1-p) = 3$ (a constant horizontal line, since both rows give the same value 3 in this column).

Plotting all three lines, the lower envelope (A's guaranteed payoff, since B picks whichever line is lowest at each p) is highest exactly where line B1 ($4-2p$) crosses line B3 (the constant 3): solving $4-2p = 3$ gives $p = 0.5$.

At $p = 0.5$, value = 3 (both crossing lines give exactly 3 there, and line B2 = $1+4(0.5) = 3$ also passes through the very same point in this particular example, confirming all three strategies meet at the optimum).

Conclusion: A's optimal mixed strategy is $p = 0.5$ for A1 and 0.5 for A2 (i.e., $(1/2, 1/2)$); the value of the game $V = 3$. Since strategy B2's line passes through the same optimal point here, B's optimal mixed strategy can be confirmed by solving the reduced 2×2 game between any two of the crossing columns (e.g., B1 and B3) using the algebraic method of Section 7.6.1.

Note: In examination problems, the lower (or, for the column player, upper) envelope will usually have its optimum at the intersection of exactly two lines; always reduce to that 2×2 sub-game and finish with the exact algebraic formulas rather than reading approximate values off the graph.

Key Takeaways — Unit 7: Game Theory

- ✓ Game Theory analyses competitive situations between rational decision-makers whose payoffs depend on each other's choices; this unit focuses on two-person, zero-sum games.
- ✓ A game is represented by a payoff matrix (from Player A's point of view); since the game is zero-sum, B's payoff is simply the negative of A's.
- ✓ The Maximin (best of A's row-minimums) and Minimax (best of B's column-maximums) criteria give each player's cautious guaranteed value; if Maximin = Minimax, a saddle point exists and the game has a pure-strategy solution.
- ✓ The Rule of Dominance lets us delete rows/columns that are never better than another row/column, shrinking the payoff matrix before solving.
- ✓ When no saddle point exists, players must use mixed strategies (randomising over multiple strategies with fixed probabilities); 2×2 games are solved directly using the algebraic formulas for p , q , and the value of the game V .
- ✓ $2 \times n$ (or $m \times 2$) games without a saddle point are solved using the graphical method, which finds the highest point on the lower envelope of B's expected-payoff lines and reduces the problem to an equivalent 2×2 sub-game.

— End of Unit 7 —