

Operations Research

Unit 2: Linear Programming Problem

(10 Hours)

1. Introduction to Linear Programming

Linear Programming (LP) is a very powerful mathematical technique used to find the **best possible decision** (maximum profit or minimum cost) when we have **limited resources**. The word '**Linear**' means all the relationships in the problem are straight-line relationships — if you double the input, the output also doubles exactly.

Real-life example: A factory makes two products using machines and labour. Both resources are limited. LP tells us exactly how many units of each product to make to earn the most profit without exceeding any resource limit.

Every LP problem has exactly four parts:

- **Decision Variables ($x_1, x_2, \dots$):** The unknown quantities we want to find. For example: how many tables and chairs to produce per week.
- **Objective Function (Z):** The mathematical goal — either MAXIMISE (profit/revenue) or MINIMISE (cost/time). Example: Maximise $Z = 700x_1 + 400x_2$
- **Constraints:** The limitations on resources, written as linear inequalities. Example: $4x_1 + 3x_2 \leq 240$ (cannot use more than 240 hours of carpentry time)
- **Non-negativity Conditions:** We cannot produce a negative quantity, so $x_1 \geq 0, x_2 \geq 0$ always.

☞ Note: LP is called 'programming' not because of computers but because 'programming' means planning or scheduling in old British English.

2. LP Problem Formulation

Formulation means converting a word problem into mathematical equations. This is the most important skill in LP. Follow these four steps every time:

Step	What to Do	How to Do It
Step 1	Identify and define decision variables	What are the quantities we control? Name them x_1 , x_2 , etc.
Step 2	Write the Objective Function	What do we want to maximise or minimise? Write it as $Z = c_1x_1 + c_2x_2$.
Step 3	Write all Constraints	Every resource limitation or requirement becomes an inequality ($\leq$, $\geq$, or $=$).
Step 4	Add Non-negativity Conditions	Always write $x_1 \geq 0$, $x_2 \geq 0$ at the end.

Example 2.1 — Furniture Company (Product Mix Problem)

Problem Statement:

A furniture company manufactures Tables and Chairs.

- Each **Table** requires: 4 hrs Carpentry + 2 hrs Finishing. Profit per table = **Rs. 700**
- Each **Chair** requires: 3 hrs Carpentry + 1 hr Finishing. Profit per chair = **Rs. 400**
- Resources available per week: **240 hrs Carpentry** and **100 hrs Finishing**.
- **Question:** How many Tables and Chairs should be produced to MAXIMIZE weekly profit?

Formulation:

Step 1 — Decision Variables:

Let x_1 = Number of Tables produced per week

Let x_2 = Number of Chairs produced per week

Step 2 — Objective Function:

Maximise $Z = 700x_1 + 400x_2$ (Total weekly profit in Rs.)

Step 3 — Constraints:

Carpentry constraint: $4x_1 + 3x_2 \leq 240$ (total carpentry hrs used $\leq$ available 240 hrs)

Finishing constraint: $2x_1 + x_2 \leq 100$ (total finishing hrs used $\leq$ available 100 hrs)

Step 4 — Non-negativity: $x_1 \geq 0$, $x_2 \geq 0$ (cannot produce negative quantities)

Example 2.2 — Diet Problem (Minimisation)

Problem Statement:

A dietitian wants to prepare a diet using two foods: Food A and Food B.

- Food A costs Rs. 2/unit and provides: 1 unit Protein, 2 units Calcium
- Food B costs Rs. 3/unit and provides: 2 units Protein, 1 unit Calcium
- Daily minimum requirements: At least 6 units Protein and 8 units Calcium
- **Question:** What quantities of Food A and Food B minimise the total cost?

Formulation:

Let x_1 = units of Food A, x_2 = units of Food B

Minimise $Z = 2x_1 + 3x_2$ (total cost)

Subject to:

Protein: $x_1 + 2x_2 \geq 6$ (must meet minimum protein)

Calcium: $2x_1 + x_2 \geq 8$ (must meet minimum calcium)

Non-negativity: $x_1 \geq 0, x_2 \geq 0$

3. Types of Constraints and Standard Form

LP constraints come in three types. To solve LP algebraically using the Simplex method, all constraints must be converted to **equalities (=)** by adding extra variables. This is called the **Standard Form**.

Type	Symbol	Common In	What Variable to Add	Example
Less than or equal	$\leq$	Resource availability (maximum)	Add SLACK variable (+s). Slack = unused resource.	$3x_1 + 2x_2 \leq 46$
Greater than or equal	$\geq$	Minimum requirements (minimum)	Subtract SURPLUS (-s), then add ARTIFICIAL (+A) for initial solution.	$x_1 + 2x_2 \geq 16$
Exactly equal	$=$	Fixed quantity required	Add ARTIFICIAL variable (+A) only.	$x_1 + x_2 = 10$

Note: A Slack Variable represents idle/unused capacity. If $s_1=10$, it means 10 hours of carpentry are unused. At the optimal solution, binding constraints have $s=0$.

4. Graphical Analysis of LP

The graphical method is used when there are **only two decision variables** (x_1 and x_2). We draw the constraints on a graph (x_1 on horizontal axis, x_2 on vertical axis) and find the **region where ALL constraints are satisfied**. This is called the **Feasible Region**.

Key terms you must know:

- **Feasible Region:** The shaded area on the graph where every single constraint is satisfied simultaneously. Any point inside this region is a valid production plan.
- **Corner Points (Extreme Points / Vertices):** The corners of the feasible region polygon. VERY IMPORTANT: The LP theorem guarantees that the optimal solution ALWAYS lies at one of these corner points.
- **Iso-profit / Iso-cost Line:** A line where $Z = \text{constant value}$. For maximisation, we push this line outward (away from origin) until it just touches the last corner of the feasible region — that corner is the optimal point.
- **Binding Constraint:** At the optimal point, a constraint that is satisfied with equality (used completely, no slack). It 'binds' or 'limits' the solution.
- **Non-binding Constraint:** A constraint with leftover slack at the optimal point. Relaxing it (increasing the RHS) would not improve Z .

☞ Note: The feasible region of any LP is always a CONVEX shape (no dents inward). This is why the optimal solution is always at a corner.

5. Graphical LP Solution — Complete Step-by-Step

The 5-Step Graphical Procedure:

Step 1: Plot Constraint Lines

Replace each inequality with $=$ and draw the straight line. Find two points: set $x_1=0$ to get x_2 -intercept, and set $x_2=0$ to get x_1 -intercept. Connect them.

Step 2: Shade Feasible Side

For $\leq$ constraints: shade the region below/left of the line (toward the origin). For $\geq$ constraints: shade above/right of the line (away from origin). The feasible region is where ALL shaded areas overlap, staying in the first quadrant ($x_1 \geq 0, x_2 \geq 0$).

Step 3: Find Corner Points

Identify all vertices of the feasible region polygon. Each corner is where two boundary lines intersect. Solve those two equations simultaneously to find exact coordinates.

Step 4: Evaluate Z at Each Corner

Substitute each corner point (x_1, x_2) into the objective function $Z = c_1x_1 + c_2x_2$ and calculate Z.

Step 5: Choose Optimal Corner

For MAXIMISATION: choose the corner with the HIGHEST Z value. For MINIMISATION: choose the corner with the LOWEST Z value.

Example 2.3 — Complete Graphical Solution (Furniture Problem from Ex. 2.1)

Given: Maximise $Z = 700x_1 + 400x_2$

Constraints: $4x_1 + 3x_2 \leq 240$ (Carpentry) and $2x_1 + x_2 \leq 100$ (Finishing), $x_1, x_2 \geq 0$

Step 1: Find intercepts of each constraint line

Line 1 ($4x_1 + 3x_2 = 240$): Set $x_1 = 0 \rightarrow x_2 = 80$. Set $x_2 = 0 \rightarrow x_1 = 60$. Points: (0,80) and (60,0)

Line 2 ($2x_1 + x_2 = 100$): Set $x_1 = 0 \rightarrow x_2 = 100$. Set $x_2 = 0 \rightarrow x_1 = 50$. Points: (0,100) and (50,0)

Step 2: Identify the Feasible Region

Both constraints are $\leq$ type, so shade below both lines. The feasible region is bounded by O(0,0), A(50,0), B(intersection), and C(0,80).

Step 3: Find Corner Point B (intersection of both constraint lines)

Solve simultaneously:

$$4x_1 + 3x_2 = 240 \text{ ... equation (i)}$$

$$2x_1 + x_2 = 100 \text{ ... equation (ii)}$$

$$\text{From (ii): } x_2 = 100 - 2x_1 \text{ ... (iii)}$$

$$\text{Substitute (iii) into (i): } 4x_1 + 3(100 - 2x_1) = 240$$

$$\Rightarrow 4x_1 + 300 - 6x_1 = 240 \Rightarrow -2x_1 = -60 \Rightarrow x_1 = 30$$

$$\text{Substitute } x_1 = 30 \text{ into (ii): } 2(30) + x_2 = 100 \Rightarrow x_2 = 40 \Rightarrow x_2 = 40$$

$\therefore$ Corner point B = (30, 40)

Step 4 & 5: Evaluate Z at all corner points

Corner Point	x_1	x_2	$Z = 700x_1 + 400x_2$	Remark
O (Origin)	0	0	$700(0) + 400(0) = 0$	Worst
A	50	0	$700(50) + 400(0) = 35,000$	Good
B (Intersection)	30	40	$700(30) + 400(40) = 37,000$	★ MAXIMUM

Corner Point	x_1	x_2	$Z = 700x_1 + 400x_2$	Remark
C	0	80	$700(0) + 400(80) = 32,000$	Good

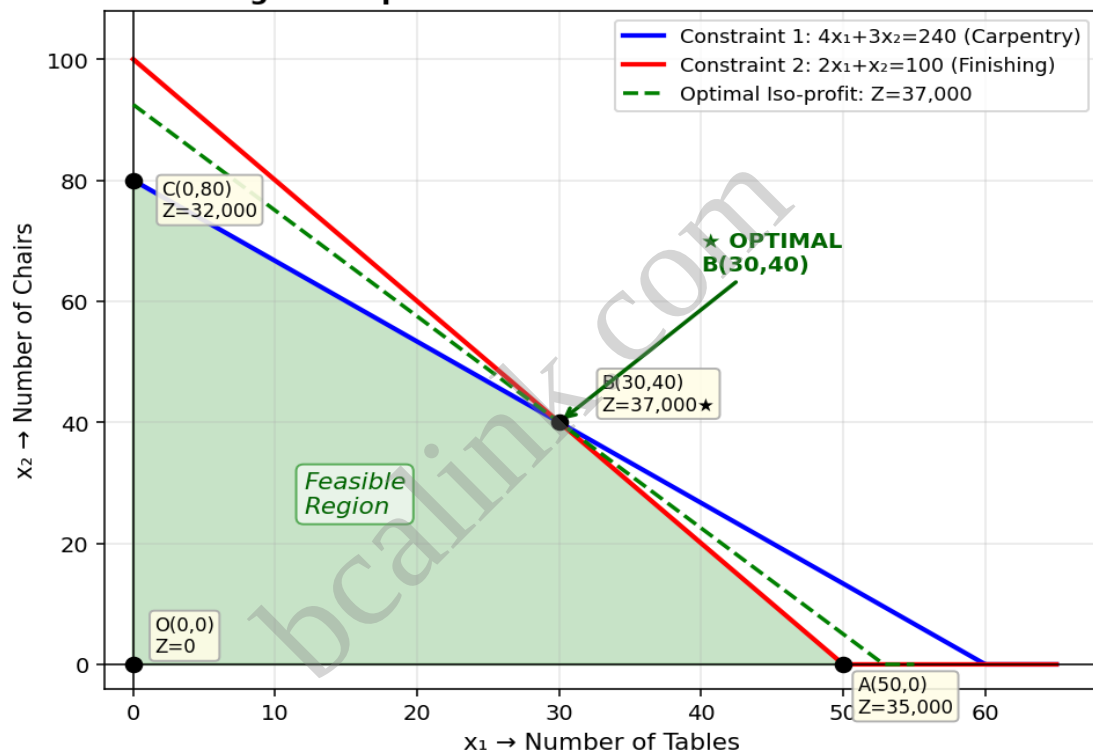
✓ **Solution:**

Optimal Solution: Produce **30 Tables** ($x_1=30$) and **40 Chairs** ($x_2=40$)

Maximum Weekly Profit: **$Z = \text{Rs. } 37,000$**

Both constraints are **BINDING** (no slack) — all carpentry and finishing hours are used up.

Fig. 1: Graphical LP Solution — Furniture Problem



The green shaded polygon is the feasible region. All four corner points (O , A , B , C) are marked with their Z values. Point $B(30,40)$ gives the highest profit. The dashed green iso-profit line $Z=37,000$ passes exactly through the optimal point B .

Example 2.4 — Minimisation: Diet Problem (Complete Graphical Solution)

Given: Minimise $Z = 2x_1 + 3x_2$

Constraints: $x_1 + 2x_2 \geq 6$ (Protein) and $2x_1 + x_2 \geq 8$ (Calcium), $x_1, x_2 \geq 0$

Step 1: Find intercepts

Line 1 ($x_1 + 2x_2 = 6$): (0,3) and (6,0)

Line 2 ($2x_1 + x_2 = 8$): (0,8) and (4,0)

Step 2: Both constraints are $\geq$ type $\rightarrow$ feasible region is ABOVE both lines.

Step 3: Find intersection point B

$x_1 + 2x_2 = 6$... (i) and $2x_1 + x_2 = 8$... (ii)

From (i): $x_1 = 6 - 2x_2$. Substitute: $2(6 - 2x_2) + x_2 = 8 \Rightarrow 12 - 4x_2 + x_2 = 8 \Rightarrow -3x_2 = -4 \Rightarrow x_2 = 4/3$

Then $x_1 = 6 - 2(4/3) = 6 - 8/3 = 10/3$

Corner B = $(10/3, 4/3) \approx (3.33, 1.33)$

Step 4 & 5: Evaluate Z

A(0,8): $Z = 2(0) + 3(8) = 24$

B($10/3, 4/3$): $Z = 2(10/3) + 3(4/3) = 20/3 + 12/3 = 32/3 \approx 10.67 \leftarrow \text{MINIMUM}$

C(4,0): $Z = 2(4) + 3(0) = 8$

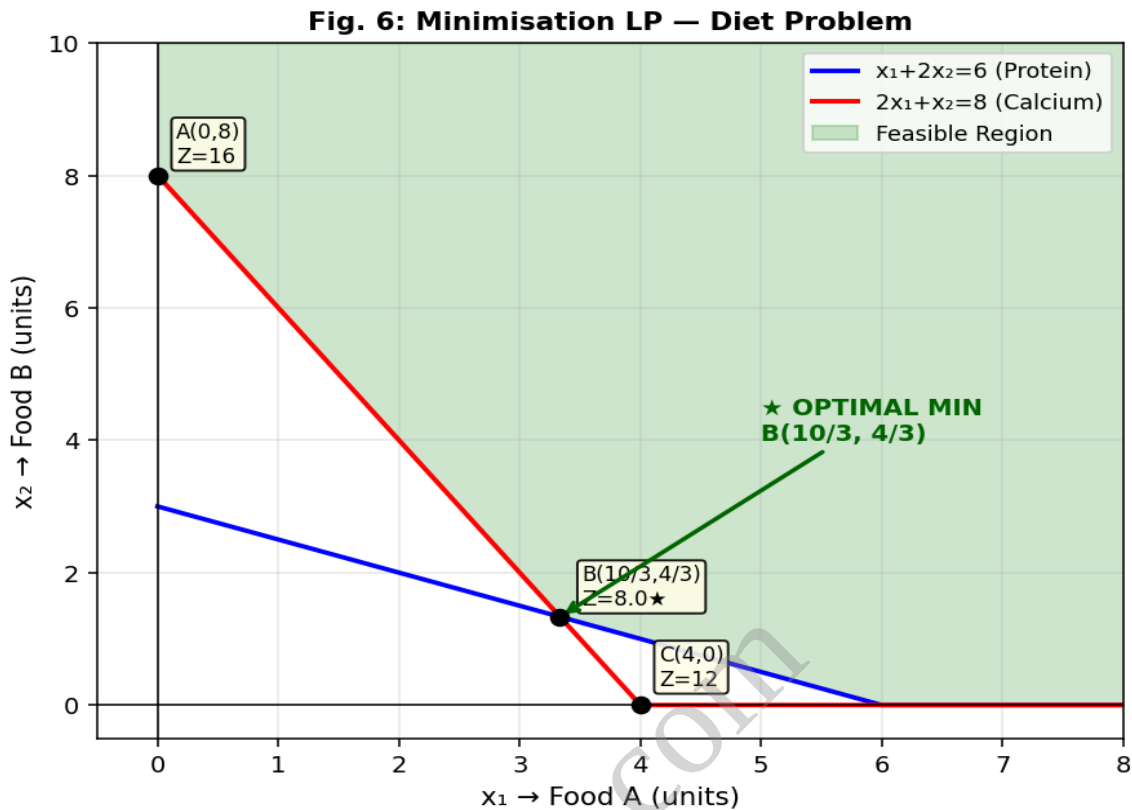
Wait — C(4,0): Check if it satisfies constraint 1: $x_1 + 2x_2 = 4 + 0 = 4 < 6$. NOT satisfied! So C is NOT feasible.

Recheck: Feasible corners are A(0,8) and B($10/3, 4/3$) only (C(4,0) violates protein constraint).

Solution:

Optimal (Minimum Cost) Solution: Food A = $10/3 \approx 3.33$ units, Food B = $4/3 \approx 1.33$ units

Minimum Daily Cost: $Z = \text{Rs. } 32/3 \approx \text{Rs. } 10.67$



For minimisation, the feasible region extends away from the origin (above both $\geq$ constraint lines). The optimal minimum cost point $B(10/3, 4/3)$ is at the inner corner of the feasible region.

6. Special Cases in Graphical LP

When solving LP graphically, we sometimes encounter special situations. You must be able to recognise and explain each one.

6.1 Multiple Optimal Solutions

This happens when the **iso-profit/iso-cost line is parallel to one of the constraint lines**. In this case, **TWO** corner points give the same maximum (or minimum) Z . Every point on the line segment connecting those two corners is also optimal. **Managerial meaning:** You have flexibility — several production plans are equally good.

Example 2.5 — Multiple Optimal Solutions

Maximise $Z = 4x_1 + 2x_2$

Subject to: $2x_1 + x_2 \leq 10$, $x_1 + x_2 \leq 6$, $x_1, x_2 \geq 0$

Corner Points:

O(0,0): $Z = 4(0) + 2(0) = 0$

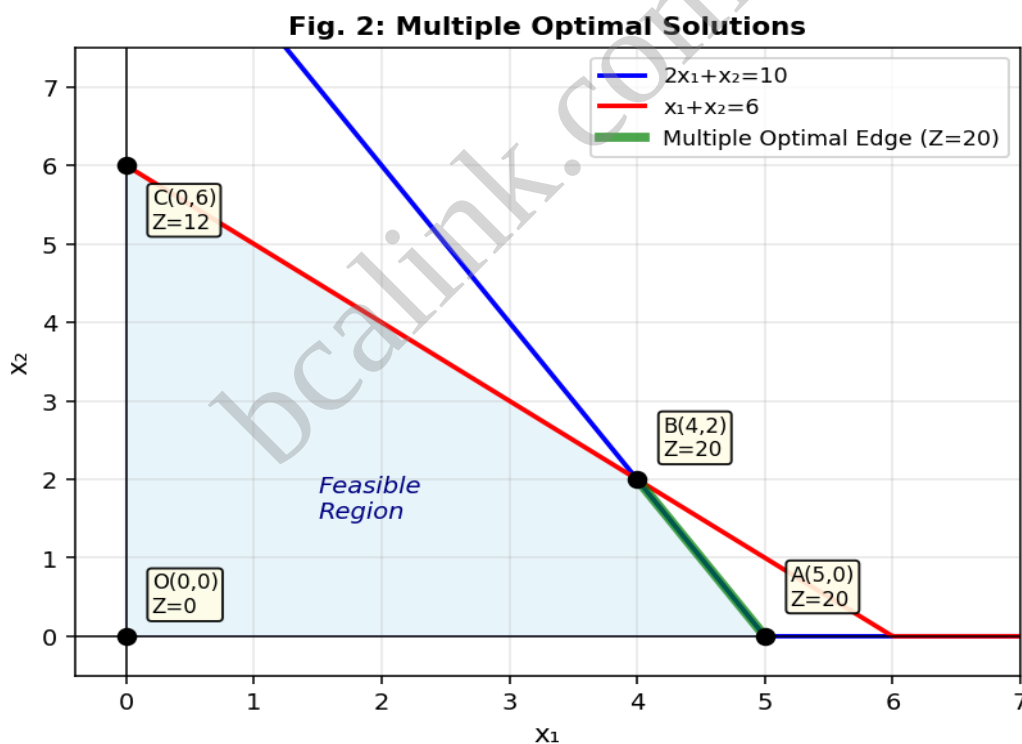
A(5,0): $Z = 4(5) + 2(0) = 20$

B(4,2): $Z = 4(4) + 2(2) = 16 + 4 = 20 \leftarrow$ Same as A!

C(0,6): $Z = 4(0) + 2(6) = 12$

Why? Notice: Objective function slope = $4/2 = 2$. Constraint 1 slope = $2/1 = 2$. They are PARALLEL!

$\therefore$ The entire segment from A(5,0) to B(4,2) is optimal. Infinitely many solutions exist with $Z=20$.



The thick green line between A(5,0) and B(4,2) shows that ALL points on that edge are optimal with $Z=20$. The iso-profit line for $Z=20$ lies exactly on top of edge AB.

6.2 Unbounded Solution

An LP is **unbounded** when the feasible region has no boundary in the direction we are optimising. We can keep increasing Z (for maximisation) forever. This is unrealistic and

means a **constraint is missing** from the model. The problem is incorrectly formulated.

Example 2.6 — Unbounded Solution

Maximise $Z = 3x_1 + 2x_2$

Subject to: $x_1 - x_2 \leq 4$, $x_1, x_2 \geq 0$

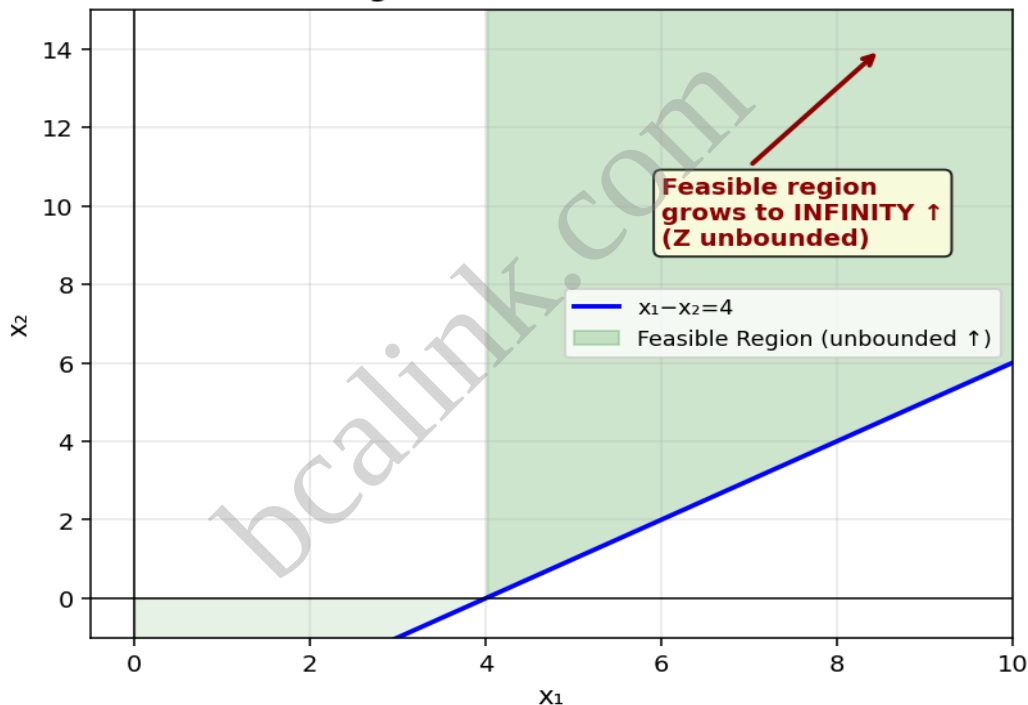
The line $x_1 - x_2 = 4$ limits x_1 from below but does NOT restrict x_2 from growing upward.

As $x_2 \rightarrow \infty$, $Z = 3x_1 + 2x_2 \rightarrow \infty$ as well. There is NO maximum value of Z .

Diagnosis: A missing constraint such as $x_1 + x_2 \leq M$ (total production limit) must be added.

Conclusion: The problem has an UNBOUNDED solution — no finite optimal exists.

Fig. 3: Unbounded Solution



The green feasible region extends infinitely upward. No matter how high you push the iso-profit line, it always touches the feasible region — so Z can grow without limit.

6.3 Infeasible Solution

An LP is **infeasible** when the constraints **contradict each other**. No point in the graph can satisfy ALL constraints at the same time. The feasible region is empty. This means the problem as stated is impossible and the constraints must be revised.

Example 2.7 — Infeasible Solution

Maximise $Z = 5x_1 + 4x_2$

Subject to: $x_1 + x_2 \leq 4$ AND $x_1 + x_2 \geq 6$, $x_1, x_2 \geq 0$

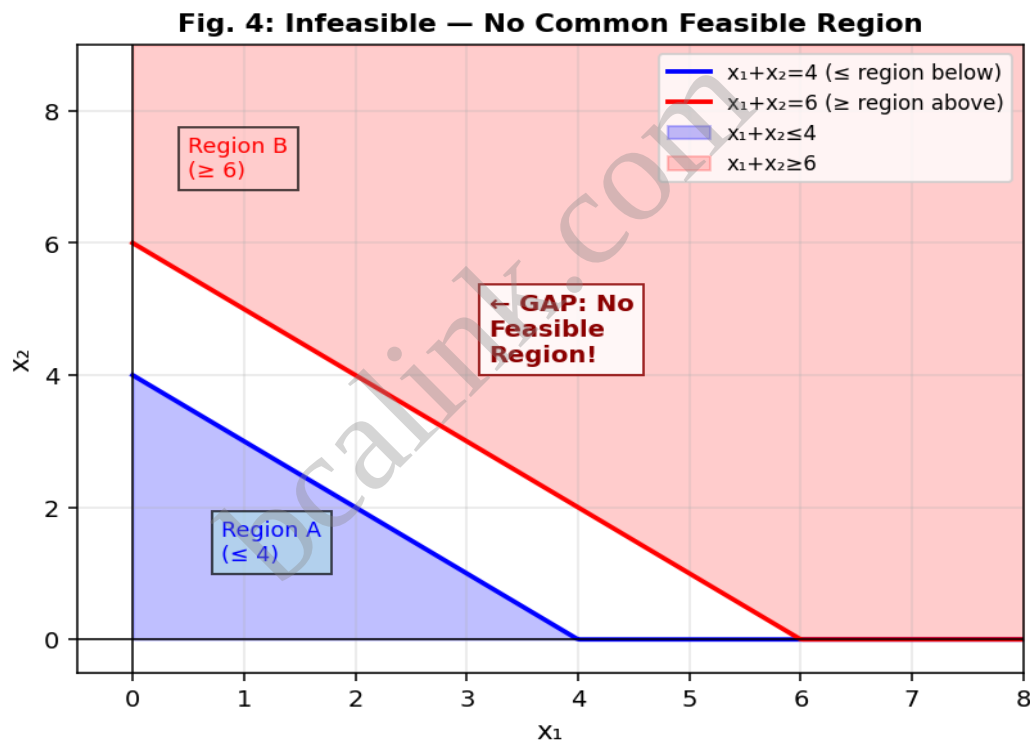
Constraint 1 says: $x_1 + x_2$ must be AT MOST 4 (≤ 4)

Constraint 2 says: $x_1 + x_2$ must be AT LEAST 6 (≥ 6)

These two constraints are CONTRADICTIONARY — no number can be both ≤ 4 AND ≥ 6 simultaneously!

Conclusion: The problem is INFEASIBLE — no feasible region exists. The LP has NO solution.

What to do? Review and correct the constraints. Perhaps the limits were entered wrongly.



The blue region satisfies $x_1 + x_2 \leq 4$ and the red region satisfies $x_1 + x_2 \geq 6$. There is a GAP between them — no point lies in both regions. The feasible region is empty.

7. Basics of the Simplex Method

The graphical method only works for **2 variables**. Real-world LP problems have hundreds or thousands of variables. The **Simplex Method** (developed by George Dantzig, 1947) is an algebraic procedure that can handle ANY number of variables.

Simple idea: The optimal solution is always at a corner point. The Simplex method cleverly moves from one corner to a BETTER neighbouring corner, step by step, until it cannot improve anymore. It is like walking uphill, always taking the steepest step, until you reach the peak.

Important terms (explained simply):

- **Basic Variable:** A variable currently IN the solution (has a non-zero value). We call it 'in the basis'. Initially, slack variables are basic.
- **Non-basic Variable:** A variable currently SET TO ZERO (not being used). Initially, decision variables x_1 and x_2 are non-basic.
- **Basic Feasible Solution (BFS):** A solution at a corner point of the feasible region. Each iteration of Simplex moves from one BFS to another.
- **Slack Variable (s):** Added to $\leq$ constraints to make them equalities. It represents unused capacity.
- **Artificial Variable (A):** Added temporarily to $\geq$ or $=$ constraints to create an initial basic feasible solution. Removed during solving.
- **Pivot Column:** The column of the variable that ENTERS the basis (selected by the most negative Z-row coefficient in maximisation).
- **Pivot Row:** The row of the variable that LEAVES the basis (selected by the minimum ratio test).
- **Pivot Element:** The number at the intersection of the pivot column and pivot row. We make it 1 using row operations.
- **Optimality Condition:** For maximisation: optimal when ALL coefficients in the Z-row are ≥ 0 (no more improvement possible).

☞ Note: The Simplex method only visits corner points — it never wastes time on interior points. For a problem with m constraints and n variables, there are at most $C(n,m)$ corner points, but Simplex usually reaches optimal in far fewer steps.

Converting to Standard Form (for $\leq$ constraints in maximisation):

Before applying Simplex, every $\leq$ constraint gets a slack variable:

- $4x_1 + 3x_2 \leq 240$ becomes $4x_1 + 3x_2 + s_1 = 240$ (s_1 = unused carpentry hours)
- $2x_1 + x_2 \leq 100$ becomes $2x_1 + x_2 + s_2 = 100$ (s_2 = unused finishing hours)
- Objective: $Z - 5x_1 - 4x_2 = 0$ (rearranged with Z on left, all vars moved to left)

☞ Note: Initial solution: Set $x_1=0$, $x_2=0$ (produce nothing). Then $s_1=240$, $s_2=100$, $Z=0$. This is the starting corner point $O(0,0)$.

8. Simplex Method — Complete Worked Example

Procedure — 7 Steps of the Simplex Algorithm:

Step 1: Write in Standard Form

Add slack variables to all $\leq$ constraints. Rewrite objective as $Z - c_1x_1 - c_2x_2 = 0$.

Step 2: Set Up Initial Tableau

Arrange all equations in a table (Simplex tableau). Slack variables form the initial basis with $Z=0$.

Step 3: Identify Pivot Column (Entering Variable)

Look at the bottom Z-row. Find the MOST NEGATIVE coefficient. That column's variable enters the basis. (Most negative = it improves Z the most per unit.)

Step 4: Minimum Ratio Test (Leaving Variable)

For each row, divide the RHS value (b) by the POSITIVE value in the pivot column. The row with the SMALLEST positive ratio gives the leaving variable. (This prevents going outside the feasible region.)

Step 5: Perform Row Operations (Pivot)

Divide the pivot row by the pivot element to make it 1. Then use row operations to make ALL other entries in the pivot column = 0. Update the basis.

Step 6: Update Z-row

Apply the same row operations to the Z-row.

Step 7: Check Optimality

If ALL Z-row coefficients are $\geq 0 \rightarrow$ OPTIMAL. Stop. Otherwise, go back to Step 3.

Example 2.8 — Full Simplex Solution

Maximise $Z = 5x_1 + 4x_2$

Subject to: $6x_1 + 4x_2 \leq 24$, $x_1 + 2x_2 \leq 6$, $x_1, x_2 \geq 0$

STEP 1: Standard Form (add slack variables s_1 and s_2)

$6x_1 + 4x_2 + s_1 = 24 \rightarrow$ Basis: $s_1 = 24$

$x_1 + 2x_2 + s_2 = 6 \rightarrow$ Basis: $s_2 = 6$

$Z - 5x_1 - 4x_2 = 0 \rightarrow$ Initial $Z = 0$

STEP 2: Initial Tableau (Starting at corner O: $x_1=0$, $x_2=0$, $s_1=24$, $s_2=6$, $Z=0$)

Basis	x_1	x_2	s_1	s_2	b (RHS)
s_1	6	4	1	0	24
s_2	1	2	0	1	6
Z	-5	-4	0	0	$0 \leftarrow Z=0$

Z-row has -5 (x_1) and -4 (x_2). Most negative = -5, so x_1 is the entering variable (pivot column = x_1 column).

STEP 3 & 4: Minimum Ratio Test

Row	Basis	RHS (b)	Pivot Col Value	Ratio = $b \div$ Pivot	Decision
Row 1	s_1	24	6	$24 \div 6 = 4 \leftarrow$ minimum	★ Pivot Row (s_1 leaves)
Row 2	s_2	6	1	$6 \div 1 = 6$	Larger, not selected

Pivot Element = 6 (Row 1, x_1 column). x_1 enters, s_1 leaves.

STEP 5: Row Operations — Iteration 1

New Row 1 = Old Row 1 $\div$ 6 (make pivot element = 1):

R1_new: $[6 \div 6, 4 \div 6, 1 \div 6, 0 \div 6, 24 \div 6] = [1, 2/3, 1/6, 0, 4] \rightarrow x_1=4$

New Row 2 = Old Row 2 - 1 $\times$ New Row 1 (eliminate x_1 from Row 2):

R2_new: $[1-1, 2-2/3, 0-1/6, 1-0, 6-4] = [0, 4/3, -1/6, 1, 2]$

New Z-row = Old Z-row + 5 $\times$ New Row 1 (eliminate x_1 from Z-row, coefficient was -5):

Z_new: $[0, -4+10/3, 0+5/6, 0, 0+20] = [0, -2/3, 5/6, 0, 20]$

Tableau after Iteration 1 (Now at corner A: $x_1=4$, $x_2=0$, $Z=20$)

Basis	x_1	x_2	s_1	s_2	b (RHS)
x_1	1	$2/3$	$1/6$	0	4
s_2	0	$4/3$	$-1/6$	1	2
Z	0	$-2/3$	$5/6$	0	$20 \leftarrow Z=20$

Z-row still has $-2/3$ (x_2 column). Not optimal yet. x_2 enters next (most negative is $-2/3$).

STEP 6: Iteration 2 — Minimum Ratio Test for x_2 entering

Row	Basis	RHS	x_2 col value	Ratio	Decision
Row 1	x_1	4	$2/3$	$4 \div (2/3) = 6$	Larger
Row 2	s_2	2	$4/3$	$2 \div (4/3) = 1.5 \leftarrow \min$	★ Pivot Row (s_2 leaves)

Pivot Element = $4/3$ (Row 2, x_2 column). x_2 enters, s_2 leaves.

New Row 2 = Old Row 2 $\times (3/4)$ (make pivot = 1):

R2_new: $[0, 1, -1/8, 3/4, 3/2] \rightarrow x_2 = 3/2 = 1.5$

New Row 1 = Old Row 1 $- (2/3) \times$ New Row 2 (eliminate x_2 from Row 1):

R1_new: $[1, 0, 1/4, -1/2, 3] \rightarrow x_1 = 3$

New Z-row = Old Z-row $+ (2/3) \times$ New Row 2:

Z_new: $[0, 0, 3/4, 1/2, 21] \rightarrow Z = 21$

Final Optimal Tableau (After Iteration 2)

Basis	x_1	x_2	s_1	s_2	b (RHS)
x_1	1	0	$1/4$	$-1/2$	3
x_2	0	1	$-1/8$	$3/4$	$3/2$
Z	0	0	$3/4$	$1/2$	$21 \leftarrow Z=21$ OPTIMAL

Z-row: coefficients of $s_1=+3/4$ and $s_2=+1/2$. ALL are ≥ 0 . No more improvement possible. STOP — OPTIMAL!

✓ **Solution:**

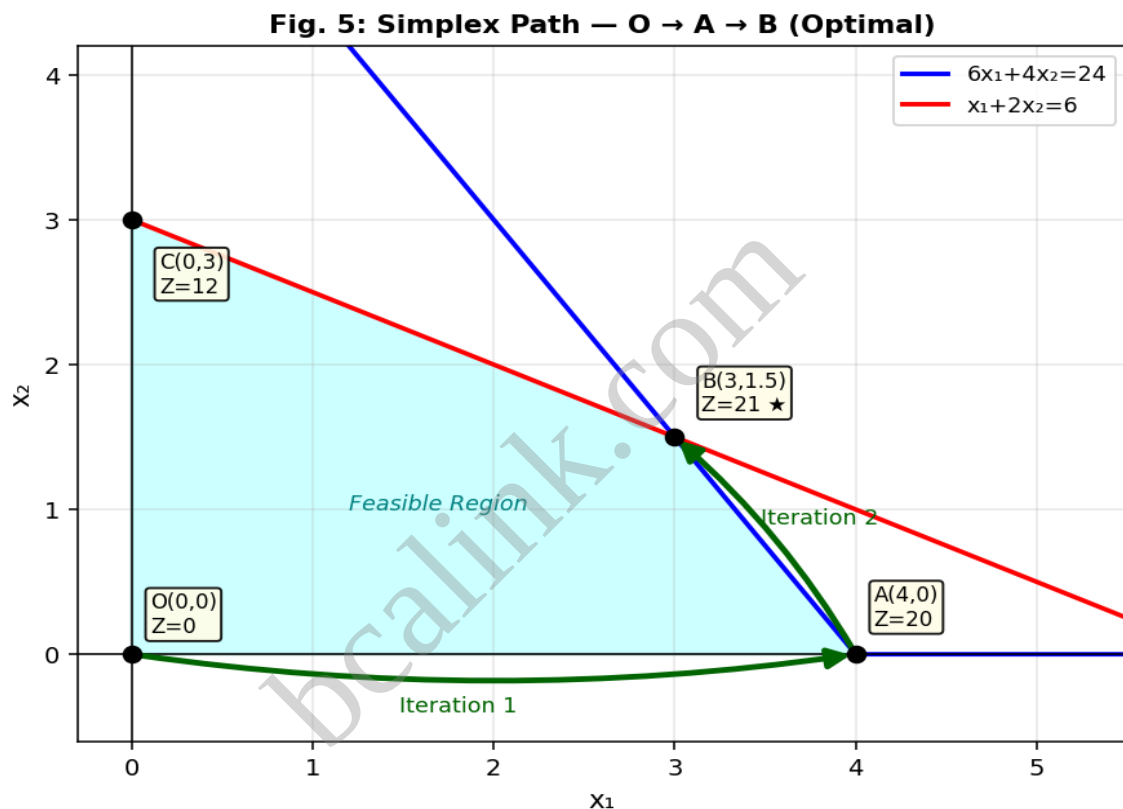
$x_1 = 3$ (produce 3 units of Product 1)

$x_2 = 1.5$ (produce 1.5 units of Product 2)

Maximum $Z = 21$

$s_1 = 0$ (Constraint 1 is binding — all of Resource 1 used up)

$s_2 = 0$ (Constraint 2 is binding — all of Resource 2 used up)



The Simplex method moves from $O(0,0) \rightarrow A(4,0) \rightarrow B(3,1.5)$. Each move improves Z . At B , no further improvement is possible — this is the optimal corner.

9. Simplex with More Than Two Variables

The power of the Simplex method is that it works for ANY number of variables. The tableau just becomes wider. Here is a 3-variable example:

Example 2.9 — Three-Variable Simplex Problem

Maximise $Z = 3x_1 + 2x_2 + 5x_3$

Subject to:

$x_1 + 2x_2 + x_3 \leq 430$...(Material constraint)

$3x_1 + 2x_3 \leq 460$...(Machine hours constraint)

$x_1 + 4x_2 \leq 420$...(Labour constraint)

$x_1, x_2, x_3 \geq 0$

Standard Form: Add s_1, s_2, s_3 to each constraint:

$x_1 + 2x_2 + x_3 + s_1 = 430, 3x_1 + 2x_3 + s_2 = 460, x_1 + 4x_2 + s_3 = 420$

$Z - 3x_1 - 2x_2 - 5x_3 = 0$

Initial Basis: $s_1=430, s_2=460, s_3=420, Z=0$

Iteration 1: Z-row: $-3(x_1), -2(x_2), -5(x_3)$. Most negative = $-5 \rightarrow x_3$ enters

Ratios: R1: $430/1=430$, R2: $460/2=230 \leftarrow$ minimum, R3: x_3 not in R3 (coeff=0, skip)

$\rightarrow s_2$ leaves. Pivot element = 2 (Row 2, x_3 column)

New R2 = $R2 \div 2$: $[3/2, 0, 1, 0, 1/2, 0, 230]$

New R1 = $R1 - 1 \times \text{NewR2}$: $[-1/2, 2, 0, 1, -1/2, 0, 200]$

New Z = $Z + 5 \times \text{NewR2}$: $[15/2 - 3, -2, 0, 0, 5/2, 0, 1150] \rightarrow Z=1150$

Continue iterations until all Z-row coefficients ≥ 0 ...

Final Optimal Solution: $x_1=0, x_2=100, x_3=230$

$Z = 3(0) + 2(100) + 5(230) = 0 + 200 + 1150 = 1,350$

✓ Solution:

Produce: $x_1=0$ units of Product 1, $x_2=100$ units of Product 2, $x_3=230$ units of Product 3

Maximum Profit: $Z = 1,350$

10. Primal and Dual Problems

Every LP problem (called the **Primal**) has a related problem called the **Dual**. The dual is automatically formed from the primal using simple rules. This is one of the most beautiful mathematical results in LP:

- If the Primal is a Maximisation problem $\rightarrow$ the Dual is a Minimisation problem (and vice versa).

- **Strong Duality Theorem:** If the primal has an optimal solution, so does the dual, AND both optimal values are EQUAL: Primal $Z^* = \text{Dual } W^*$.
- **Complementary Slackness:** At optimality, if a primal constraint has slack > 0 , its dual variable $= 0$. If a dual constraint has surplus > 0 , its primal variable $= 0$.

Rules for Converting Primal (Max) $\rightarrow$ Dual (Min):

Primal (Maximisation)	Dual (Minimisation)
Maximise $Z = c_1x_1 + c_2x_2 + \dots + c_nx_n$	Minimise $W = b_1y_1 + b_2y_2 + \dots + b_my_m$
m constraints (each $\leq$)	m dual variables ($y_1, y_2, \dots, y_m$), each ≥ 0
n decision variables	n dual constraints (each $\geq$)
Objective coefficients $c \rightarrow$ RHS of dual constraints	RHS values $b \rightarrow$ Objective coefficients of dual
Constraint matrix $A \rightarrow$ Transposed (A^T) in dual	If primal constraint i is $\leq \rightarrow$ dual variable $y_i \geq 0$

Example 2.10 — Primal to Dual Conversion (Step by Step)

PRIMAL Problem:

Maximise $Z = 5x_1 + 4x_2$

Subject to: $6x_1 + 4x_2 \leq 24$...(Constraint 1, RHS = 24)

$x_1 + 2x_2 \leq 6$...(Constraint 2, RHS = 6)

$x_1, x_2 \geq 0$

Step-by-Step Conversion to DUAL:

1. Primal has 2 constraints → Dual has 2 variables: y_1 (for constraint 1) and y_2 (for constraint 2)

2. Primal has 2 variables → Dual has 2 constraints

3. Dual objective uses RHS of primal as coefficients: $W = 24y_1 + 6y_2$

4. Dual constraint for x_1 : use column 1 of $A = [6, 1]$ → $6y_1 + y_2 \geq 5$ (RHS = obj. coeff of $x_1 = 5$)

5. Dual constraint for x_2 : use column 2 of $A = [4, 2]$ → $4y_1 + 2y_2 \geq 4$ (RHS = obj. coeff of $x_2 = 4$)

DUAL Problem:

Minimise $W = 24y_1 + 6y_2$

Subject to: $6y_1 + y_2 \geq 5$

$4y_1 + 2y_2 \geq 4$

$y_1, y_2 \geq 0$

Verification (Strong Duality):

We know primal optimal is $Z=21$ ($x_1=3, x_2=1.5$).

Solve dual: From final simplex tableau Z-row, dual variables are $y_1=3/4, y_2=1/2$

$W = 24(3/4) + 6(1/2) = 18 + 3 = 21$ ✓ (Same as primal optimal — Strong Duality confirmed!)

11. Economic Interpretation of LP and Duality

The LP solution gives managers much more than just the optimal quantities. It provides **rich economic information** about the value of resources, the cost of restrictions, and the stability of the solution.

Shadow Price (Dual Variable y_i):

The shadow price of constraint i tells a manager: 'If I get ONE additional unit of this resource, by exactly how much will the optimal Z improve?' It is the marginal value of a resource. In Example 2.10: $y_1=3/4$ means one extra hour of Resource 1 increases profit by Rs. 0.75.

Binding vs Non-binding Constraints:

A BINDING constraint (slack=0) uses all available resource. Its shadow price > 0 — worth acquiring more. A NON-BINDING constraint (slack >0) has surplus. Its shadow price $= 0$ — getting more would NOT help.

Reduced Cost:

For a variable not in the optimal solution (non-basic, value=0), the reduced cost = amount by which its objective coefficient must IMPROVE before it becomes worth producing. In Example 2.8: non-basic variables are s_1 and s_2 with Z -row values $3/4$ and $1/2$.

Sensitivity Analysis (Ranging):

Examines: 'How much can we change a parameter (cost coefficient or resource limit) before the OPTIMAL BASIS changes?' This gives managers a safe operating range. For example: 'Profit per table can drop from Rs.700 to Rs.X before our production plan should change.'

Example 2.11 — Shadow Price Interpretation

From Example 2.8: Optimal solution: $x_1=3$, $x_2=1.5$, $Z=21$

Shadow prices from final tableau Z-row: $y_1 = 3/4 = 0.75$, $y_2 = 1/2 = 0.50$

Interpretation:

- Resource 1 shadow price = 0.75: If Resource 1 increases from 24 $\rightarrow$ 25 units, Z increases by 0.75 (from 21 to 21.75).
- Resource 2 shadow price = 0.50: If Resource 2 increases from 6 $\rightarrow$ 7 units, Z increases by 0.50 (from 21 to 21.50).
- Both constraints are BINDING ($s_1=0$, $s_2=0$) $\rightarrow$ both resources are fully used.

Managerial Decision:

If the company can purchase ONE more unit of additional capacity, it should buy Resource 1 (shadow price $0.75 > 0.50$) as it gives a higher return per unit.

Maximum amount worth paying for one extra unit of Resource 1 = Rs. 0.75.

Strong Duality Verification: $W = 24(0.75) + 6(0.50) = 18 + 3 = 21 = Z \checkmark$

Unit 2 — Complete Summary / Key Takeaways

1. **LP** optimises a linear objective function (Z) subject to linear constraints using decision variables.
2. **Formulation steps:** Define variables → Write objective → Write constraints → Add non-negativity.
3. **Constraint types:** $\leq$ (add slack +s), $\geq$ (subtract surplus -s, add artificial +A), $=$ (add artificial +A).
4. **Graphical method** (2 variables only): Plot constraints → Find feasible region → Evaluate Z at ALL corner points → Pick best corner.
5. **Special cases:**
 - Multiple Optimal: iso-profit line parallel to a constraint edge; entire edge is optimal.
 - Unbounded: feasible region open in direction of optimisation; a constraint is missing.
 - Infeasible: constraints contradict each other; no feasible region exists.
6. **Simplex Method:** Start at $O(0,0)$, move to better corners via pivot operations. Stop when Z-row has no negative values.
7. **Simplex steps:** Standard form → Initial tableau → Find pivot column (most negative Z) → Ratio test (min ratio) → Row operations → Repeat.
8. **Dual problem:** Mirror of primal. Strong Duality: Primal $Z^* =$ Dual W^* always.
9. **Shadow prices** = marginal value of each resource = dual variable values from final tableau.
10. **Sensitivity analysis:** How much can parameters change before the optimal basis changes?