

UNIT 6

Permutation and Combination

(7 Hours) — Mathematics

This unit introduces the fundamental counting techniques used throughout combinatorics, probability, and statistics. We begin with the basic principle of counting and build up to permutations of distinct and repeated objects, circular arrangements, and combinations, along with the key algebraic properties that connect them.

1. Basic Principle of Counting

1.1 Fundamental Principle of Multiplication

If an operation can be performed in m different ways, and following it, a second independent operation can be performed in n different ways, then the two operations together can be performed in $m \times n$ different ways. This extends to any number of successive operations: if there are k stages with $n_1, n_2, \dots, n_k$ ways respectively, the total number of ways to complete all stages is $n_1 \times n_2 \times \dots \times n_k$.

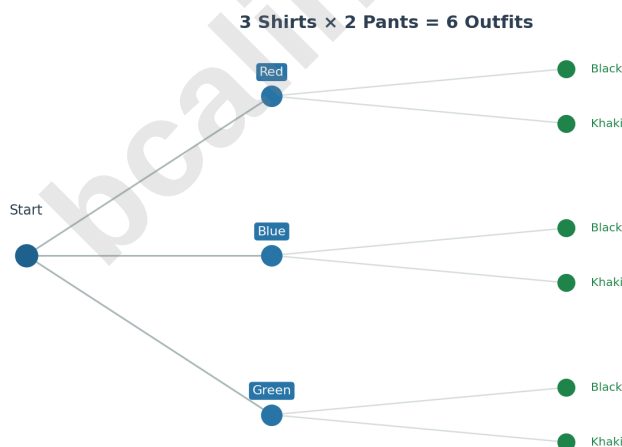


Figure 1: Tree diagram showing the multiplication principle — 3 shirt choices $\times$ 2 pant choices = 6 possible outfits.

Example 1.1

A person has 3 shirts and 2 pairs of trousers. In how many ways can they choose an outfit consisting of one shirt and one pair of trousers?

Solution

By the fundamental principle of multiplication:

Number of ways = $3 \times 2 = 6$ ways.

1.2 Fundamental Principle of Addition

If an operation can be performed in m ways and a second, mutually exclusive operation can be performed in n ways, then either operation (but not both together) can be performed in $m + n$ ways. This principle is used when the choices represent alternative, non-overlapping paths rather than successive stages.

Example 1.2

A student can choose a project from 4 topics in Mathematics or 5 topics in Computer Science, but not both. In how many ways can the student choose a single project topic?

Solution

By the principle of addition:

Number of ways = $4 + 5 = 9$ ways.

Note: Use multiplication when tasks happen in sequence (AND); use addition when choosing between mutually exclusive alternatives (OR).

2. Permutation of a Set of Objects All Different

A permutation is an arrangement of objects in a definite order. When all n objects are distinct, the number of ways to arrange r of them ($r \leq n$) in a row, where order matters, is denoted ${}^n P_r$ and given by:

Formula

$${}^n P_r = n! / (n - r)!$$

where $n! = n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1$, and $0! = 1$ by convention.

When $r = n$, every object is used and:

Special Case

$${}^n P_n = n! / 0! = n!$$

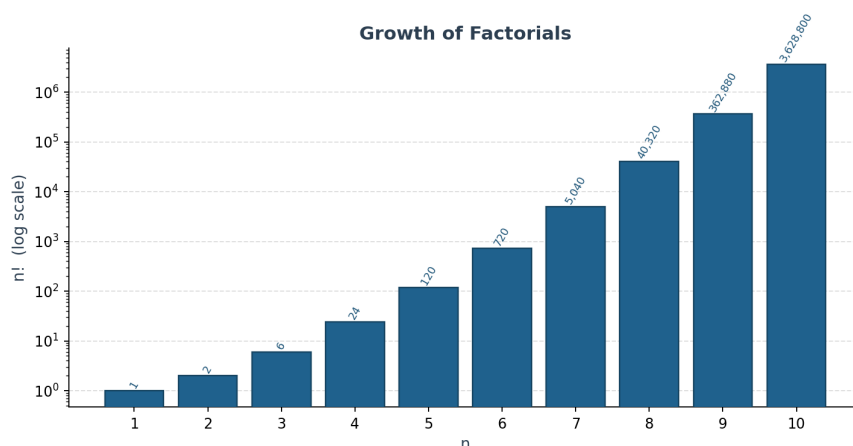


Figure 2: Growth of $n!$ for $n = 1$ to 10 , plotted on a logarithmic scale — factorials grow extremely fast.

Example 2.1

In how many ways can 3 books be arranged out of 5 different books on a shelf?

Solution

Here $n = 5$, $r = 3$.

$${}^5P_3 = 5! / (5-3)! = 5! / 2! = 120 / 2 = 60 \text{ ways.}$$

Example 2.2

Find the number of ways of arranging the letters of the word 'MOUSE' taken all at a time.

Solution

All 5 letters are distinct, so $n = 5$, $r = 5$.

$${}^5P_5 = 5! = 120.$$

There are 120 distinct arrangements.

3. Permutation of a Set of Objects Not All Different

When some objects in the set are alike (indistinguishable from one another), interchanging identical objects does not create a new arrangement. If n objects consist of p objects of one kind, q objects of a second kind, r objects of a third kind, and so on ($p + q + r + \dots = n$), the number of distinct permutations of all n objects is:

Formula

$$\text{Number of arrangements} = n! / (p! \times q! \times r! \times \dots)$$

Example 3.1

Find the number of distinct arrangements of the letters of the word 'STATISTICS'.

Solution

The word STATISTICS has 10 letters: S occurs 3 times, T occurs 3 times, I occurs 2 times, A occurs 1 time, C occurs 1 time.

$$\begin{aligned}\text{Number of arrangements} &= 10! / (3! \times 3! \times 2! \times 1! \times 1!) \\ &= 3,628,800 / (6 \times 6 \times 2) = 3,628,800 / 72 = 50,400.\end{aligned}$$

Example 3.2

How many different words (not necessarily meaningful) can be formed using all the letters of the word 'BANANA'?

Solution

BANANA has 6 letters: A occurs 3 times, N occurs 2 times, B occurs 1 time.

$$\text{Number of arrangements} = 6! / (3! \times 2! \times 1!) = 720 / 12 = 60.$$

4. Circular Arrangement

When objects are arranged around a circle rather than in a row, one position must be fixed as a reference, because rotating the entire arrangement produces the same relative order. This removes one degree of freedom compared to a linear arrangement.

Formula

Number of circular arrangements of n distinct objects $= (n - 1)!$

If clockwise and anticlockwise arrangements are considered identical (e.g. arranging beads on a necklace or flowers in a garland), the count is further halved: $(n - 1)! / 2$

Circular Arrangement of 5 Objects

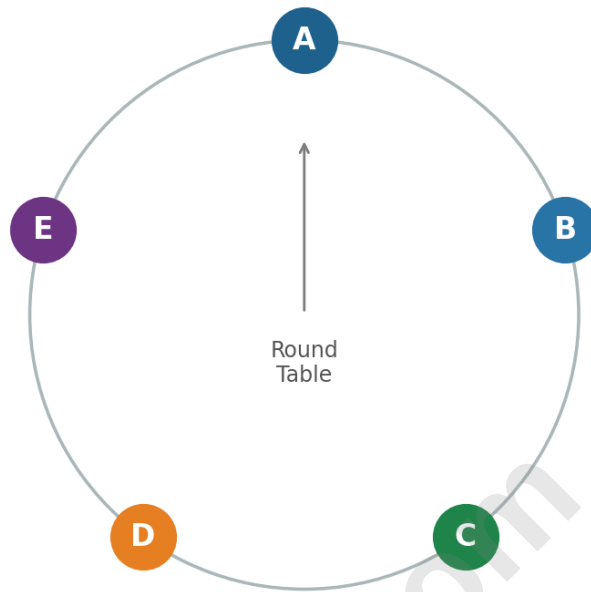


Figure 3: Circular arrangement of 5 distinct objects (A–E) around a round table; one seat is fixed as the reference point.

Example 4.1

In how many ways can 5 people be seated around a circular table?

Solution

Number of ways = $(5 - 1)! = 4! = 24$ ways.

Example 4.2

In how many ways can 6 different beads be arranged to form a necklace?

Solution

Since a necklace can be flipped over, clockwise and anticlockwise arrangements are considered the same.

Number of ways = $(6 - 1)! / 2 = 5! / 2 = 120 / 2 = 60$ ways.

Note: Distinguish carefully between problems that treat mirror-image (flippable) arrangements as identical (necklaces, garlands) and those that do not (seating arrangements around a table).

5. Repeated Use of the Same Object

So far we assumed each object could be used at most once in an arrangement. When repetition of objects is allowed — that is, the same object may be selected again for a different position — the counting changes. If there are n distinct objects and repetition is permitted, the number of ways to arrange r positions is:

Formula

Number of arrangements with repetition allowed = n^r

This is because each of the r positions can independently be filled in n ways (the fundamental principle of multiplication applied r times in succession).

Example 5.1

How many 3-digit codes can be formed using the digits 0–9 if repetition of digits is allowed?

Solution

Here $n = 10$ digits, $r = 3$ positions, repetition allowed.

Number of codes = $10^3 = 1,000$ codes.

Example 5.2

How many 4-letter words (not necessarily meaningful) can be formed from the 26 letters of the English alphabet if letters may repeat?

Solution

$n = 26$, $r = 4$, repetition allowed.

Number of words = $26^4 = 456,976$.

Note: Compare this with ${}^nP_r$, which assumes no repetition. When repetition is allowed, the count is always n^r , which grows much faster than ${}^nP_r = n!/(n-r)!$ for the same n and r .

6. Combination of Things All Different

A combination is a selection of objects where order does not matter — unlike a permutation. The number of ways of selecting r objects out of n distinct objects is denoted ${}^nC_r$ and is given by:

Formula

${}^nC_r = n! / [r! (n - r)!] = {}^nP_r / r!$

Since each combination of r objects can be arranged in $r!$ different orders (each giving a distinct permutation), dividing ${}^nP_r$ by $r!$ removes the effect of order and leaves only the number of

distinct selections.

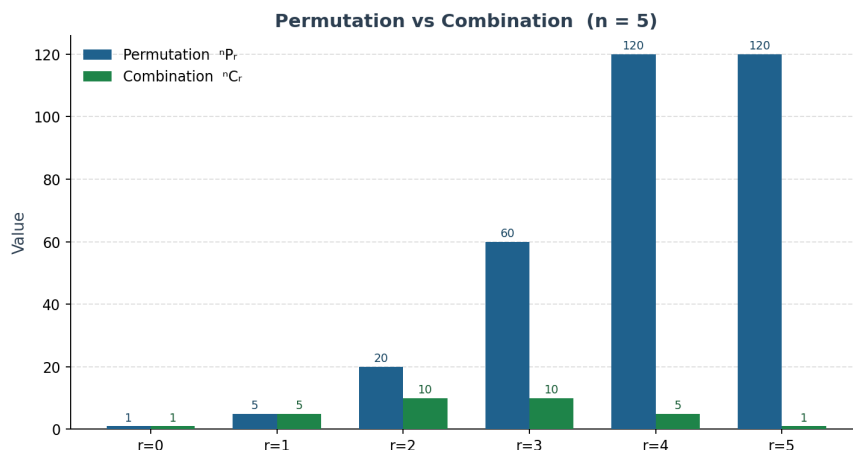


Figure 4: Comparison of ${}^n P_r$ and ${}^n C_r$ for $n = 5$ and $r = 0$ to 5 — permutations always equal or exceed combinations for the same r .

Example 6.1

In how many ways can a committee of 3 members be selected from a group of 7 people?

Solution

Here $n = 7$, $r = 3$ (order does not matter — it is a selection, not an arrangement).

$${}^7 C_3 = 7! / [3! \times 4!] = 5040 / (6 \times 24) = 5040 / 144 = 35 \text{ ways.}$$

Example 6.2

A box contains 5 red and 4 blue balls. In how many ways can 2 red and 2 blue balls be selected?

Solution

Select 2 red out of 5: ${}^5 C_2 = 5! / (2! \times 3!) = 10$.

Select 2 blue out of 4: ${}^4 C_2 = 4! / (2! \times 2!) = 6$.

By the multiplication principle: total ways = $10 \times 6 = 60$ ways.

7. Properties of Combination

The following algebraic properties of ${}^n C_r$ are frequently used to simplify combinatorial expressions and solve equations involving combinations:

Property	Statement	Meaning / Use
P1	${}^n C_r = {}^n C_{n-r}$	Choosing r objects to include is the same as choosing $(n - r)$ objects to leave out.

Property	Statement	Meaning / Use
P2	${}^nC_0 = {}^nC_n = 1$	There is exactly one way to choose nothing, and one way to choose everything.
P3	${}^nC_1 = n$	There are exactly n ways to choose a single object.
P4	${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$	Pascal's Rule — relates combinations of n objects to combinations of $(n+1)$ objects.
P5	If ${}^nC_r = {}^nC_k$ then $r = k$ or $r = n - k$	Used to solve equations of the form ${}^nC_r = {}^nC_k$ for unknown r .
P6	${}^nC_r = (n/r) \times {}^{n-1}C_{r-1}$	A recursive relation connecting successive combination values.

Example 7.1

If ${}^nC_8 = {}^nC_6$, find n .

Solution

Using Property P5: ${}^nC_r = {}^nC_k \Rightarrow r + k = n$ (when $r \neq k$).

So $8 + 6 = n \Rightarrow n = 14$.

Example 7.2

Verify Pascal's Rule for $n = 5$, $r = 2$, i.e. show that ${}^5C_2 + {}^5C_1 = {}^6C_2$.

Solution

$${}^5C_2 = 5!/(2! \times 3!) = 10.$$

$${}^5C_1 = 5.$$

$${}^5C_2 + {}^5C_1 = 10 + 5 = 15.$$

$${}^6C_2 = 6!/(2! \times 4!) = 720/48 = 15.$$

Since both sides equal 15, Pascal's Rule is verified.

Note: Pascal's Rule (P4) is the algebraic basis for constructing Pascal's Triangle, where each entry is the sum of the two entries above it.

Key Takeaways

- ✓ Multiplication principle: successive independent stages multiply ($m \times n$); Addition principle: mutually exclusive alternatives add ($m + n$).
- ✓ Permutation (order matters) of n distinct objects taken r at a time: ${}^n P_r = n! / (n - r)!$.
- ✓ When objects repeat within the set, divide by the factorial of each repeated group's count: $n! / (p! q! r! \dots)$.
- ✓ Circular arrangements of n distinct objects: $(n - 1)!$; halve this if clockwise/anticlockwise are equivalent.
- ✓ If repetition of objects is allowed across r positions: n^r .
- ✓ Combination (order does not matter): ${}^n C_r = n! / [r!(n - r)!] = {}^n P_r / r!$.
- ✓ Key properties: ${}^n C_r = {}^n C_{n-r}$, ${}^n C_0 = {}^n C_n = 1$, and Pascal's Rule ${}^n C_r + {}^n C_{r-1} = {}^{n+1} C_r$.

— End of Unit 6 —

bcalink.com