

UNIT 2

Relations and Functions

Mathematics I (CACS104)

Building on the set theory of Unit 1, this unit develops the idea of a relation between two sets and specializes it to the more restrictive, and far more widely used, idea of a function. Relations and functions are the language used to describe how one quantity depends on another, and they underlie algebra, trigonometry, calculus, and computer science alike.

1. Ordered Pairs and Cartesian Product

An ordered pair (a, b) consists of two elements a and b in a fixed order — a is called the first component and b the second component. Unlike a set, order matters: $(a, b) = (c, d)$ if and only if $a = c$ and $b = d$. In particular, $(a, b) \neq (b, a)$ unless $a = b$.

Cartesian Product

Given two non-empty sets A and B , the Cartesian product $A \times B$ is the set of all ordered pairs (a, b) where $a \in A$ and $b \in B$:

Definition

$$A \times B = \{(a, b) : a \in A \text{ and } b \in B\}$$

If $n(A) = m$ and $n(B) = n$, then $n(A \times B) = m \cdot n$. In general $A \times B \neq B \times A$ (unless $A = B$), since the order of components is not interchangeable.

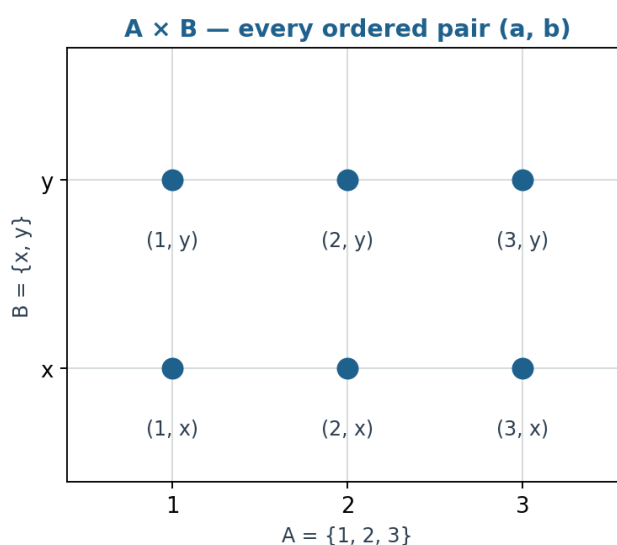


Figure 2.1 — $A \times B$ for $A = \{1, 2, 3\}$ and $B = \{x, y\}$: each grid point is one ordered pair.

Example 2.1

If $A = \{1, 2\}$ and $B = \{3, 4, 5\}$, find $A \times B$ and $n(A \times B)$.

Solution

$$A \times B = \{(1,3), (1,4), (1,5), (2,3), (2,4), (2,5)\}$$

$$n(A \times B) = n(A) \cdot n(B) = 2 \times 3 = 6$$

2. Relations

A relation R from a set A to a set B is any subset of the Cartesian product $A \times B$. If $(a, b) \in R$, we say a is related to b , and write $a R b$.

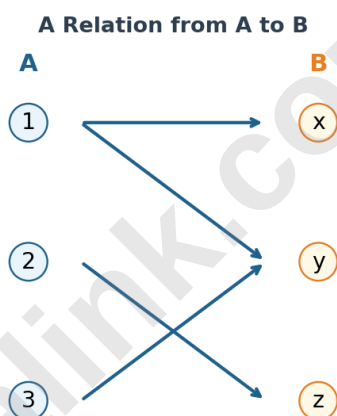


Figure 2.2 — A relation R from $A = \{1, 2, 3\}$ to $B = \{x, y, z\}$: each arrow is one ordered pair in R .

2.1 Domain and Range of a Relation

Term	Definition
Domain of R	The set of all first components of the ordered pairs in R (a subset of A)
Range of R	The set of all second components of the ordered pairs in R (a subset of B)
Codomain	The entire set B , whether or not every element of B appears in the range

2.2 Inverse of a Relation

If R is a relation from A to B , its inverse relation R^{-1} is the relation from B to A obtained by reversing every ordered pair:

Definition

$$R^{-1} = \{(b, a) : (a, b) \in R\}$$

Clearly, $\text{Domain}(R^{-1}) = \text{Range}(R)$ and $\text{Range}(R^{-1}) = \text{Domain}(R)$.

Example 2.2

Let $A = \{1, 2, 3\}$, $B = \{4, 5\}$, and $R = \{(1,4), (2,4), (3,5)\}$ be a relation from A to B .

Find: (a) Domain and Range of R (b) R^{-1}

Solution

(a) $\text{Domain}(R) = \{1, 2, 3\}$, $\text{Range}(R) = \{4, 5\}$

(b) $R^{-1} = \{(4,1), (4,2), (5,3)\}$, a relation from B to A

3. Types of Relations (on a set A)

When a relation R is defined from a set A to itself ($R \subseteq A \times A$), it may have special structural properties. The four most important are defined below and illustrated in Figure 2.3.

Property	Definition
Reflexive	$(a, a) \in R$ for every $a \in A$ — every element is related to itself
Symmetric	Whenever $(a, b) \in R$, then $(b, a) \in R$ also
Transitive	Whenever $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$ also
Equivalence	A relation that is simultaneously reflexive, symmetric, AND transitive

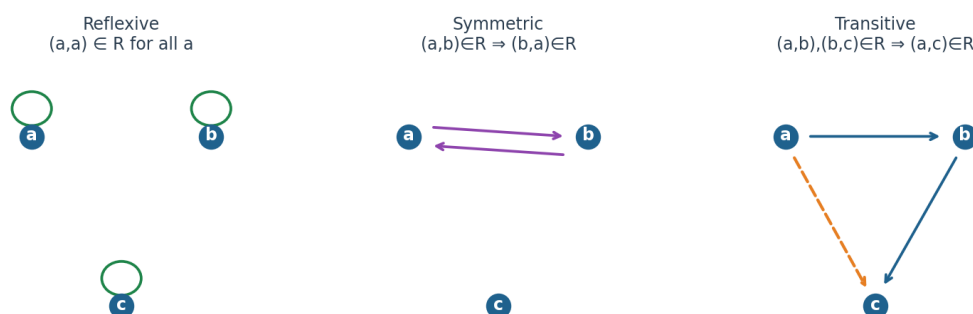


Figure 2.3 — Visual illustration of reflexive (self-loops), symmetric (two-way arrows), and transitive ($a \rightarrow b \rightarrow c$ forces $a \rightarrow c$) properties.

Note: An equivalence relation partitions the set A into disjoint groups called equivalence classes, where all elements within a class are related to each other. "Equality" ($=$) and "has the same remainder when divided by n " are classic examples of equivalence relations.

Example 2.3

On the set $A = \{1, 2, 3, 4\}$, let $R = \{(1,1), (2,2), (3,3), (4,4), (1,2), (2,1)\}$.

Determine whether R is reflexive, symmetric, and/or transitive.

Solution

Reflexive: $(1,1), (2,2), (3,3), (4,4)$ are all in $R \Rightarrow R$ is reflexive.

Symmetric: $(1,2) \in R$ and $(2,1) \in R \Rightarrow$ the symmetric condition holds here.

Transitive: $(1,2) \in R$ and $(2,1) \in R \Rightarrow$ we need $(1,1) \in R$ — true, it is.

So R is reflexive, symmetric, and transitive — hence R is an equivalence relation.

4. Definition of a Function

A function f from a set A to a set B is a relation that assigns to every element of A exactly one element of B . It is written $f : A \rightarrow B$, and for $a \in A$, the corresponding element in B is denoted $f(a)$, called the image of a under f .

The extra condition that distinguishes a function from a general relation is that no element of A is left unassigned, and no element of A is assigned to more than one element of B . Graphically, this is the vertical line test: a curve represents a function if and only if every vertical line crosses it at most once.

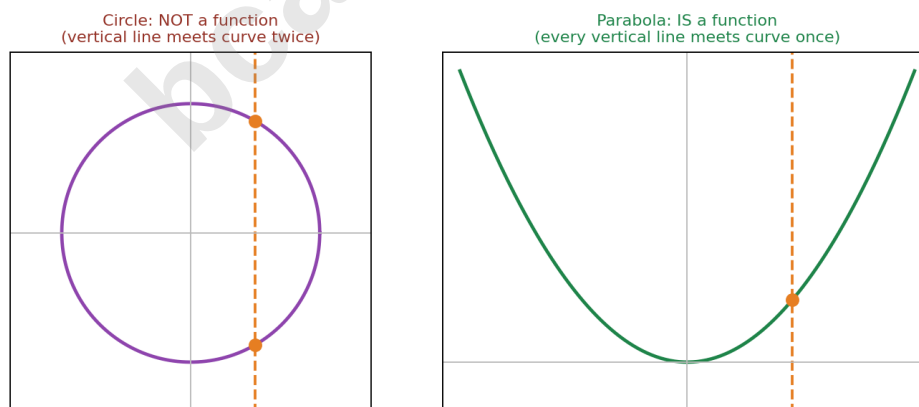


Figure 2.4 — The vertical line test: a circle fails (two y -values for one x), a parabola passes (one y -value for every x).

4.1 Domain and Range of a Function

Term	Definition
Domain	The set of all permissible input values (x -values) for which f is defined
Range	The set of all output values ($f(x)$ -values) actually produced by f

Example 2.4

Find the domain of $f(x) = 1 / (x - 2)$.

Solution

f is undefined when the denominator is 0, i.e. when $x = 2$.

Domain of $f = \mathbb{R} - \{2\} = (-\infty, 2) \cup (2, \infty)$

4.2 Inverse Function

If $f : A \rightarrow B$ is a function that is both one-to-one (injective — distinct inputs give distinct outputs) and onto (surjective — every element of B is an image of some element of A), then f has an inverse function $f^{-1} : B \rightarrow A$ satisfying:

Definition

$f^{-1}(f(x)) = x$ for all $x \in A$, $f(f^{-1}(y)) = y$ for all $y \in B$

To find $f^{-1}(x)$ algebraically: write $y = f(x)$, interchange x and y , then solve for y .

Example 2.5

Find the inverse of $f(x) = 3x + 5$.

Solution

Let $y = 3x + 5$. Interchange x and y : $x = 3y + 5$

Solve for y : $y = (x - 5)/3$

So $f^{-1}(x) = (x - 5)/3$

5. Special Functions

Function	Rule	Description
Identity function	$f(x) = x$	Maps every element to itself; domain = range = $\mathbb{R}$
Constant function	$f(x) = c$ (c fixed)	Maps every input to the same fixed output c

6. Algebraic Functions

An algebraic function is one built from polynomial expressions in x . The three basic types, in increasing order of degree, are shown below.

Type	General form	Degree	Graph shape
Linear	$f(x) = mx + c$	1	A straight line with slope m and y -intercept c
Quadratic	$f(x) = ax^2 + bx + c, a \neq 0$	2	A parabola opening upward ($a > 0$) or downward ($a < 0$)
Cubic	$f(x) = ax^3 + bx^2 + cx + d, a \neq 0$	3	An S-shaped curve with up to two turning points

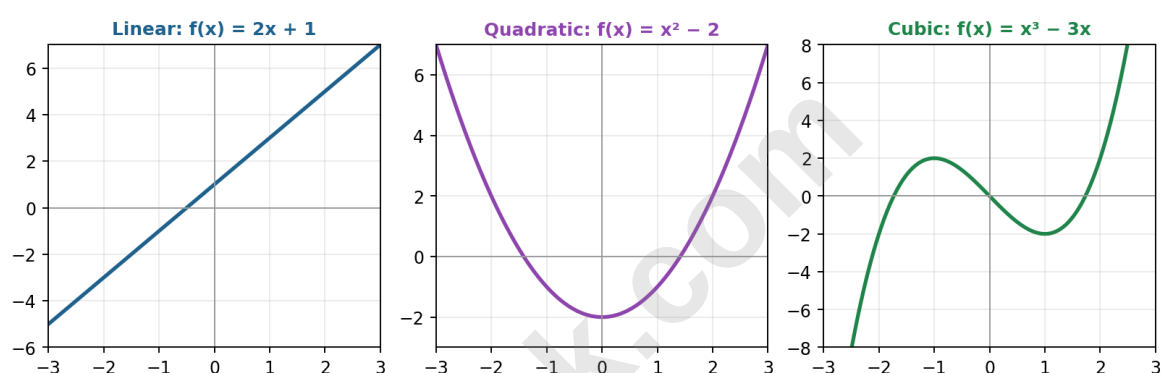


Figure 2.5 — Linear, quadratic, and cubic graphs.

Example 2.6

For $f(x) = x^2 - 2$, find $f(0)$, $f(2)$, and the values of x for which $f(x) = 0$.

Solution

$$f(0) = 0^2 - 2 = -2$$

$$f(2) = 2^2 - 2 = 2$$

$$f(x) = 0 \Rightarrow x^2 - 2 = 0 \Rightarrow x^2 = 2 \Rightarrow x = \pm\sqrt{2}$$

7. Trigonometric Functions and Their Graphs

The trigonometric functions $\sin x$, $\cos x$, and $\tan x$ are periodic functions of an angle x (measured in radians), fundamental to the study of oscillation, waves, and circular motion.

Function	Period	Range	Key feature
$\sin x$	2π	$[-1, 1]$	Passes through the origin; odd function
$\cos x$	2π	$[-1, 1]$	Equals 1 at $x = 0$; even function

Function	Period	Range	Key feature
$\tan x$	π	$(-\infty, \infty)$	Undefined (vertical asymptotes) at $x = \pi/2 + n\pi$

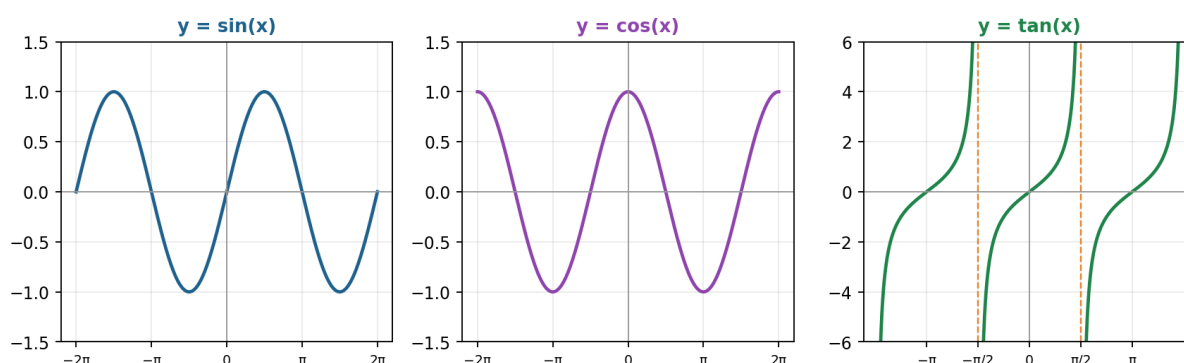


Figure 2.6 — Graphs of $y = \sin(x)$, $y = \cos(x)$, and $y = \tan(x)$, showing periodicity and the asymptotes of $\tan(x)$.

Note: $\sin x$ and $\cos x$ are defined for all real x and oscillate between -1 and 1 , while $\tan x = \sin x / \cos x$ is undefined wherever $\cos x = 0$.

8. Exponential and Logarithmic Functions

8.1 Exponential Function

An exponential function has the form $f(x) = a^x$, where $a > 0$ and $a \neq 1$ is called the base. Its domain is all of $\mathbb{R}$, its range is $(0, \infty)$, and it is always increasing when $a > 1$. The most important base in mathematics is the number $e \approx 2.71828$, giving the natural exponential function $f(x) = e^x$.

8.2 Logarithmic Function

The logarithmic function is defined as the inverse of the exponential function. For base a : $y = \log_a x$ if and only if $x = a^y$. The natural logarithm, written $\ln x$, is the logarithm with base e , and is the inverse of $f(x) = e^x$. Its domain is $(0, \infty)$ and its range is all of $\mathbb{R}$.

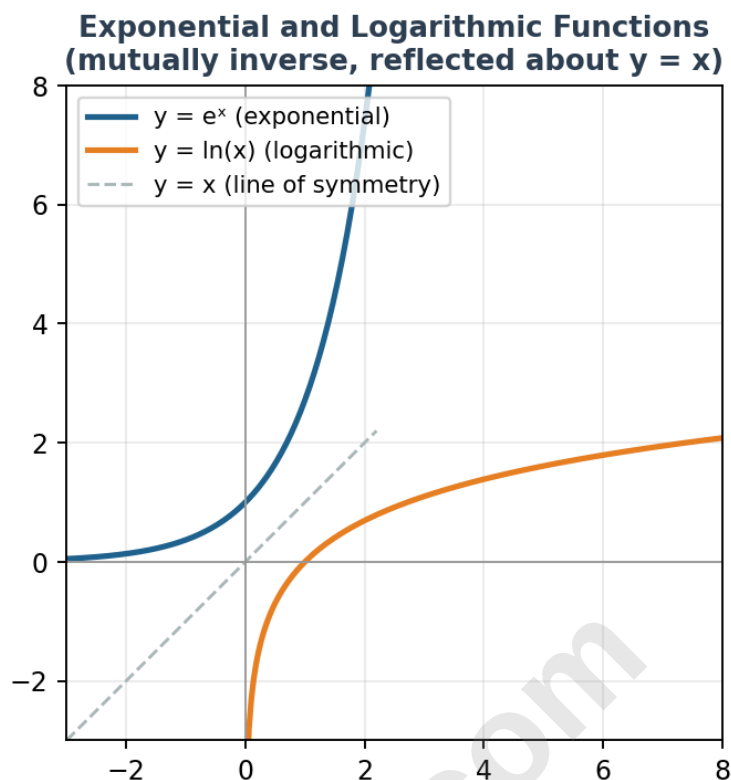


Figure 2.7 — $y = e^x$ and $y = \ln(x)$ are mirror images of each other across the line $y = x$, confirming they are inverse functions.

Example 2.7

Solve for x : $\log_2 x = 5$

Solution

$$\log_2 x = 5 \Rightarrow x = 2^5 = 32$$

9. Composite Functions

Given two functions $f : A \rightarrow B$ and $g : B \rightarrow C$, the composite function $g \circ f : A \rightarrow C$ (read "g of f" or "g composed with f") is defined by applying f first, then g to the result:

Definition

$$(g \circ f)(x) = g(f(x))$$

Note the order: $g \circ f$ means f acts first. In general, $g \circ f \neq f \circ g$.

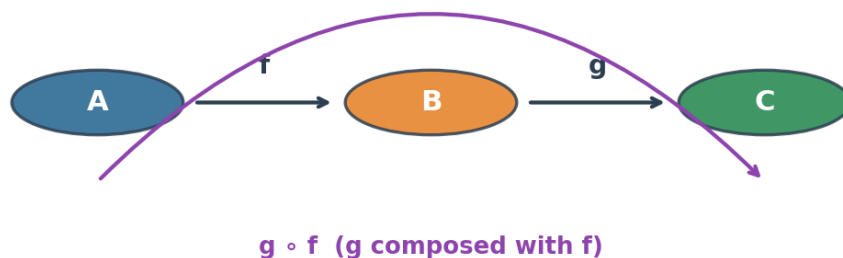


Figure 2.8 — The composite $g \circ f$: an element of A is sent to B by f , then to C by g , in a single combined step.

Example 2.8

If $f(x) = x + 3$ and $g(x) = 2x^2$, find (a) $(g \circ f)(x)$ (b) $(f \circ g)(x)$ (c) $(g \circ f)(1)$

Solution

(a) $(g \circ f)(x) = g(f(x)) = g(x + 3) = 2(x + 3)^2$

(b) $(f \circ g)(x) = f(g(x)) = f(2x^2) = 2x^2 + 3$

(c) $(g \circ f)(1) = 2(1 + 3)^2 = 2(4)^2 = 32$

Note: Parts (a) and (b) above confirm that composition is generally not commutative: $2(x+3)^2$ and $2x^2+3$ are different functions.

Key Takeaways

- ✓ An ordered pair (a, b) is order-sensitive; the Cartesian product $A \times B$ collects all ordered pairs with first element from A and second from B .
- ✓ A relation from A to B is any subset of $A \times B$; its domain, range, and inverse R^{-1} follow directly from the ordered pairs it contains.
- ✓ A relation on a set can be reflexive, symmetric, and/or transitive; one that is all three is called an equivalence relation.
- ✓ A function is a relation that assigns exactly one output to every input — checked graphically with the vertical line test.
- ✓ A one-to-one, onto function has an inverse f^{-1} , found by swapping x and y and solving for y .
- ✓ Linear, quadratic, and cubic functions are the basic algebraic function families, distinguished by degree and graph shape.
- ✓ $\sin x$, $\cos x$, and $\tan x$ are periodic; the exponential a^x and logarithm $\log_a x$ are mutual inverses, mirror images across $y = x$.
- ✓ The composite $(g \circ f)(x) = g(f(x))$ applies f first, then g ; composition is generally not commutative.

— *End of Unit 2* —

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