

UNIT 5

Analytical Geometry

(8 Hours) — Mathematics I (CACS104), Year/Sem I/I

This unit introduces conic sections — the circle, parabola, ellipse, and hyperbola — obtained by slicing a double right circular cone with a plane. We define each curve, derive its standard equation, study its graph and key terms (focus, directrix, vertex, eccentricity), and classify all conics using eccentricity. We also express these curves in polar form, which is often more convenient in physics and astronomy. The second part of the unit covers vectors in space: their algebra, length and distance formulas, and the scalar (dot) and cross products, together with their geometric meaning.

5.1 Conic Sections

A conic section is the curve obtained by intersecting a plane with a double right circular cone. Depending on the angle at which the plane cuts the cone, four distinct curves result: the circle, the ellipse, the parabola, and the hyperbola. All four can also be defined purely in terms of distances in the plane, which is the approach used throughout this unit.

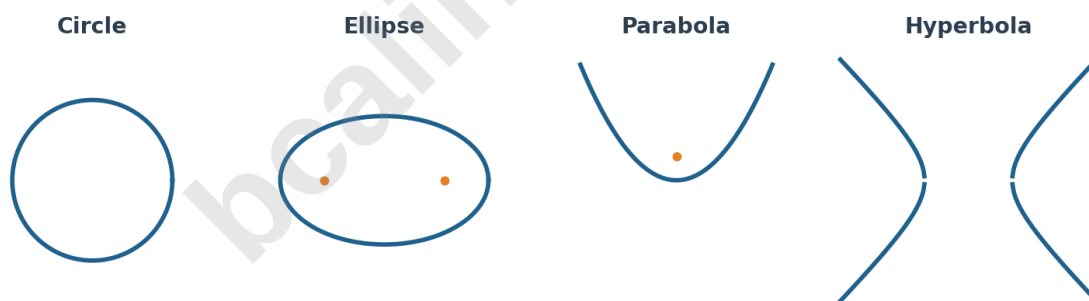


Figure 5.1 — The four conic sections

5.1.1 Definitions and Related Terms

Circle: the set of all points $P(x, y)$ in a plane that are at a constant distance r (the radius) from a fixed point $C(h, k)$ (the center).

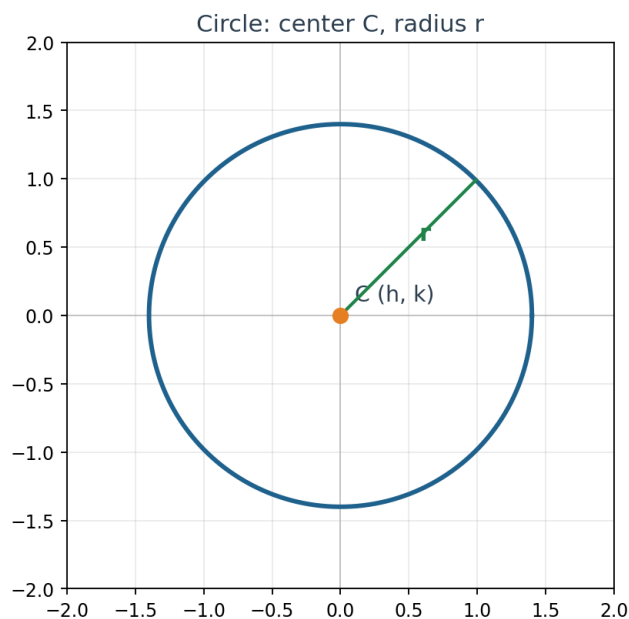


Figure 5.2 — Circle with center C and radius r

Standard equation of a circle with center (h, k) and radius r :

$$(x - h)^2 + (y - k)^2 = r^2$$

When the center is at the origin, this reduces to $x^2 + y^2 = r^2$.

Parabola: the set of all points $P(x, y)$ that are equidistant from a fixed point F (the focus) and a fixed line (the directrix).

Related terms: the vertex is the point where the parabola crosses its own axis of symmetry (the axis); the latus rectum is the chord through the focus, perpendicular to the axis, whose length is $4a$ for the standard form below.

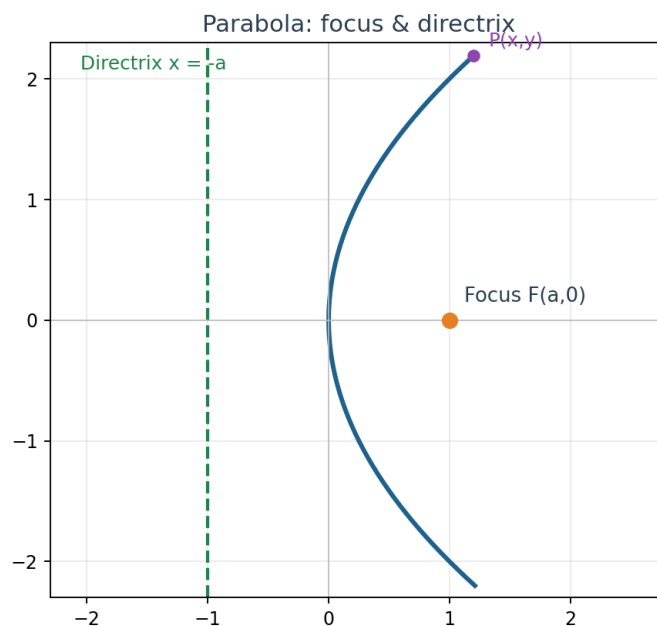


Figure 5.3 — Parabola: focus $F(a, 0)$ and directrix $x = -a$

Standard equation of a parabola with vertex at the origin, opening rightward:

$$y^2 = 4ax \quad (\text{focus at } (a, 0), \text{ directrix: } x = -a)$$

Example 5.1 — Identifying a parabola

Find the focus and directrix of the parabola $y^2 = 12x$.

Solution

Comparing with $y^2 = 4ax$ gives $4a = 12$, so $a = 3$.

Focus: $F(3, 0)$. Directrix: $x = -3$.

Length of latus rectum $= 4a = 12$.

Ellipse: the set of all points $P(x, y)$ such that the sum of the distances from P to two fixed points F_1 and F_2 (the foci) is a constant, $2a$. The line through the foci meets the ellipse at two vertices; the segment joining the vertices is the major axis (length $2a$), and the perpendicular segment through the center is the minor axis (length $2b$).

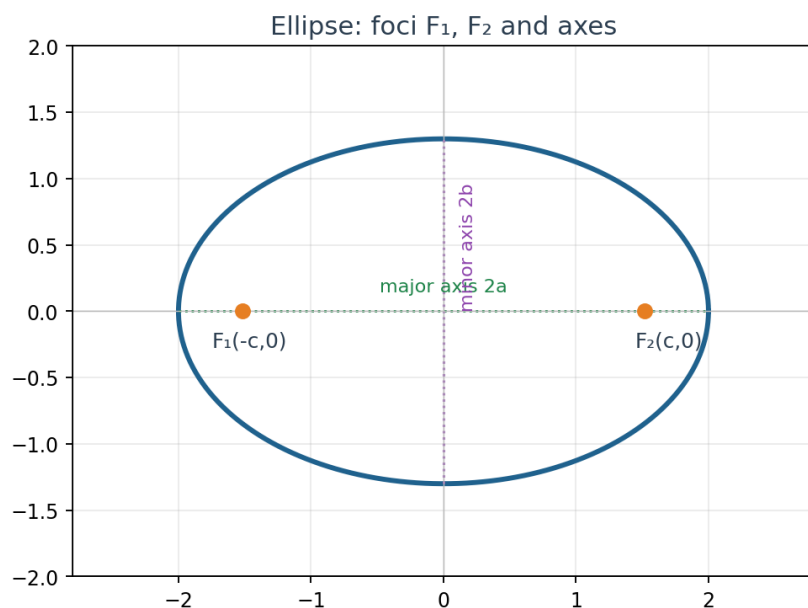


Figure 5.4 — Ellipse with foci F_1 , F_2 , major axis $2a$, minor axis $2b$

Standard equation of an ellipse centered at the origin, major axis along x:

$$x^2/a^2 + y^2/b^2 = 1 \quad (a > b, c^2 = a^2 - b^2)$$

Here c is the distance from the center to each focus, and the foci lie at $(\pm c, 0)$.

Hyperbola: the set of all points $P(x, y)$ such that the absolute difference of the distances from P to two fixed foci F_1 and F_2 is a constant, $2a$. A hyperbola has two branches, two vertices, and two asymptotes — straight lines that the branches approach but never touch.

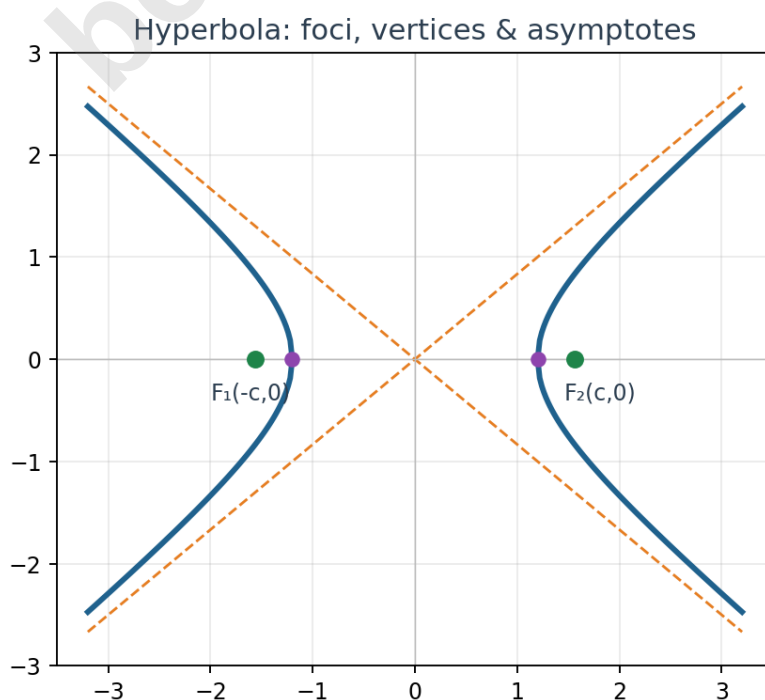


Figure 5.5 — Hyperbola with foci, vertices, and asymptotes $y = \pm(b/a)x$

Standard equation of a hyperbola centered at the origin, transverse axis along x:

$$x^2/a^2 - y^2/b^2 = 1 \quad (c^2 = a^2 + b^2)$$

The asymptotes of this hyperbola are the lines $y = (b/a)x$ and $y = -(b/a)x$.

Note: For all conics except the circle, the constant c (distance from center to focus) satisfies a simple relation with a and b — subtraction for the ellipse, addition for the hyperbola — which is worth memorizing to avoid sign errors.

Example 5.2 — Ellipse from given data

An ellipse has vertices at $(\pm 5, 0)$ and foci at $(\pm 3, 0)$. Find its equation.

Solution

Here $a = 5$, so $a^2 = 25$, and $c = 3$, so $c^2 = 9$.

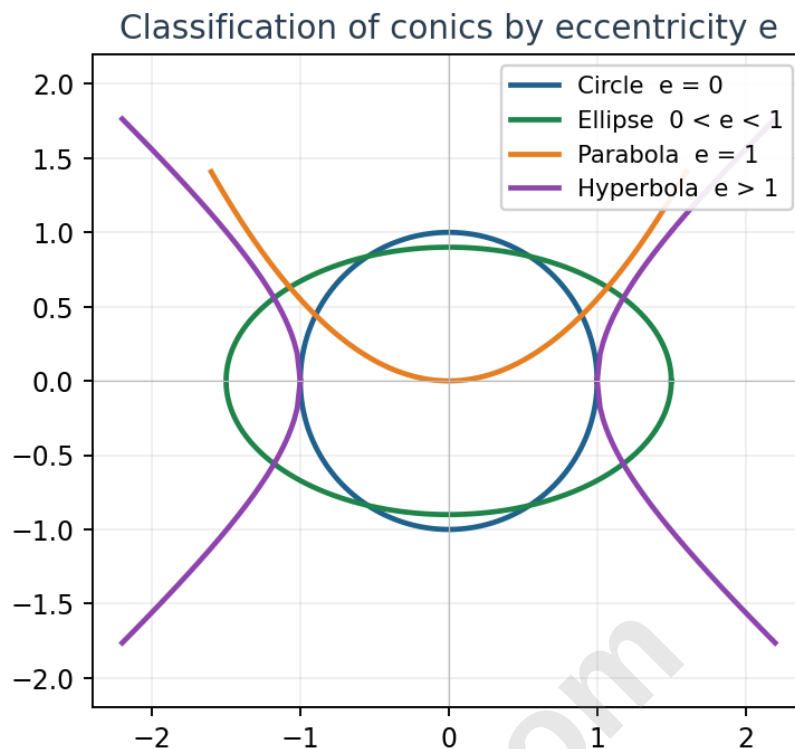
Since $b^2 = a^2 - c^2 = 25 - 9 = 16$, $b = 4$.

Equation: $x^2/25 + y^2/16 = 1$.

5.1.2 Classification of Conics by Eccentricity

Every conic (other than a circle treated as a limiting case) can be defined by a single unifying property: the ratio of the distance from a point on the curve to a fixed focus, to its distance from a fixed directrix, is a constant called the eccentricity, denoted e . This ratio classifies the conic as follows:

Conic	Eccentricity (e)	Shape
Circle	$e = 0$	Perfectly round; focus = center
Ellipse	$0 < e < 1$	Elongated closed curve
Parabola	$e = 1$	Open curve, single branch
Hyperbola	$e > 1$	Open curve, two branches



For the ellipse and hyperbola, eccentricity is computed as $e = c/a$, where c is the focus-to-center distance and a is the semi-major (or semi-transverse) axis length.

Example 5.3 — Computing eccentricity

Find the eccentricity of the hyperbola $x^2/9 - y^2/16 = 1$.

Solution

Here $a^2 = 9$, $b^2 = 16$, so $c^2 = a^2 + b^2 = 25$, $c = 5$.

Eccentricity $e = c/a = 5/3 \approx 1.67$, and since $e > 1$ this confirms the curve is a hyperbola.

5.1.3 Polar Equations of Conics

Placing a focus at the pole (origin) of a polar coordinate system gives a single unified polar equation covering all conics — circle, ellipse, parabola, and hyperbola — through one formula involving the eccentricity e and the semi-latus rectum l :

$$r = l / (1 - e \cos \theta) \quad \text{or} \quad r = l / (1 + e \cos \theta)$$

The choice of sign and whether $\cos \theta$ or $\sin \theta$ appears depends on the orientation of the directrix relative to the pole. Substituting $e = 0$, $0 < e < 1$, $e = 1$, or $e > 1$ into this single formula reproduces the circle, ellipse, parabola, and hyperbola respectively — this is the elegance of the polar approach.

Curve	Polar equation (pole at focus)	Note
Line	$r \cos \theta = k$ (or $r \sin \theta = k$)	Perpendicular distance k from pole
Circle	$r = 2a \cos \theta$ (radius a , passing through pole)	Special case $e = 0$
Ellipse / Hyperbola / Parabola	$r = l/(1 - e \cos \theta)$	l = semi-latus rectum = b^2/a (or $2a$ for parabola)

Example 5.4 — Polar equation of a parabola

A parabola has its focus at the pole and directrix at a perpendicular distance 4 units to the left of the pole. Write its polar equation.

Solution

For a parabola, $e = 1$ and the semi-latus rectum equals the focus-to-directrix distance, so $l = 4$.

Since the directrix lies to the left of the pole, the equation takes the form $r = l/(1 - \cos \theta)$.

Polar equation: $r = 4/(1 - \cos \theta)$.

Key Takeaways — Conic Sections

- ✓ A conic is the intersection of a plane with a double cone: circle, ellipse, parabola, or hyperbola.
- ✓ Circle: constant distance from center. Parabola: equidistant from focus and directrix. Ellipse: constant sum of focal distances. Hyperbola: constant difference of focal distances.
- ✓ Eccentricity e unifies all conics: $e = 0$ (circle), $0 < e < 1$ (ellipse), $e = 1$ (parabola), $e > 1$ (hyperbola).
- ✓ The polar equation $r = l/(1 - e \cos \theta)$ represents any conic with one focus at the pole.

5.2 Vectors in Space

A vector in space is a quantity possessing both magnitude and direction, represented in three-dimensional Cartesian space by an ordered triple $\mathbf{v} = (v_1, v_2, v_3)$, or equivalently $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$, where $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are the standard unit vectors along the x , y , and z axes.

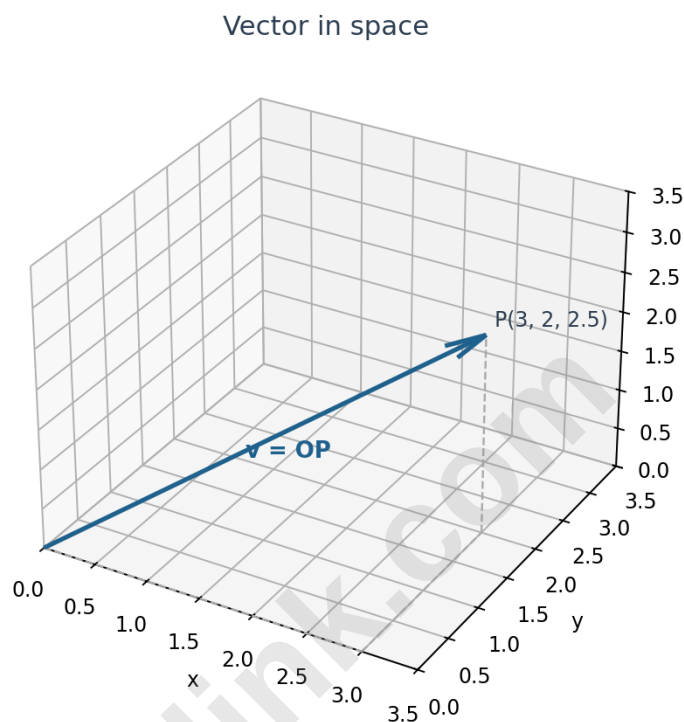


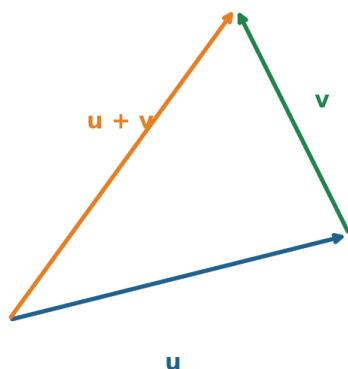
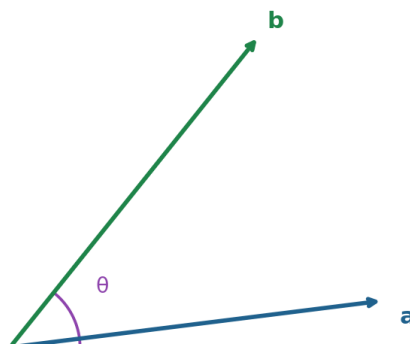
Figure 5.7 — A vector $\mathbf{v} = \overrightarrow{OP}$ in three-dimensional space

5.2.1 Algebra of Vectors in Space

Given two vectors $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$, and a scalar k , the basic operations are defined component-wise:

Operation	Definition
Addition	$\mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2, u_3 + v_3)$
Subtraction	$\mathbf{u} - \mathbf{v} = (u_1 - v_1, u_2 - v_2, u_3 - v_3)$
Scalar multiplication	$k \cdot \mathbf{u} = (ku_1, ku_2, ku_3)$
Equality	$\mathbf{u} = \mathbf{v} \Leftrightarrow u_1 = v_1, u_2 = v_2, u_3 = v_3$

Vector addition (triangle law)

Dot product: angle θ between a, b Figure 5.8 — Vector addition by the triangle law (left); angle θ between two vectors (right)

Vector addition satisfies the familiar algebraic laws: it is commutative ($u + v = v + u$), associative ($(u+v)+w = u+(v+w)$), and has an identity element, the null vector $0 = (0, 0, 0)$, such that $u + 0 = u$ for every vector u .

5.2.2 Length, Distance, Unit Vector, and Null Vector

The length (or magnitude, or norm) of a vector $v = (v_1, v_2, v_3)$ is:

$$|v| = \sqrt{v_1^2 + v_2^2 + v_3^2}$$

The distance between two points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ in space is the length of the vector AB , giving the three-dimensional distance formula:

$$d(A, B) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

A unit vector is any vector with length exactly 1. Given any nonzero vector v , its corresponding unit vector (in the same direction) is obtained by dividing v by its own length:

$$\hat{v} = v / |v|$$

The null vector (or zero vector) $0 = (0, 0, 0)$ has length 0 and no defined direction; it is the additive identity for vector addition.

Example 5.5 — Length, distance, and unit vector

Given points $A(1, 2, 2)$ and $B(4, 6, 2)$, find (a) the length of AB , (b) the distance between A and B , and (c) the unit vector in the direction of AB .

Solution

(a) $\mathbf{AB} = \mathbf{B} - \mathbf{A} = (4-1, 6-2, 2-2) = (3, 4, 0)$.

$|\mathbf{AB}| = \sqrt{(3^2 + 4^2 + 0^2)} = \sqrt{25} = 5$.

(b) Distance $d(\mathbf{A}, \mathbf{B}) = |\mathbf{AB}| = 5$ (same computation as above).

(c) Unit vector $= \mathbf{AB} / |\mathbf{AB}| = (3/5, 4/5, 0)$.

5.2.3 Scalar (Dot) Product

The scalar product (or dot product) of two vectors $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$ is defined algebraically as:

$$\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$$

and geometrically as:

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| |\mathbf{b}| \cos\theta$$

where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$. The dot product returns a scalar, not a vector — hence the name.

Geometrical interpretation: $\mathbf{a} \cdot \mathbf{b}$ represents the product of the magnitude of $\mathbf{a}$ and the projection (component) of $\mathbf{b}$ along the direction of $\mathbf{a}$. When $\mathbf{a} \cdot \mathbf{b} = 0$ (and neither vector is zero), the two vectors are perpendicular (orthogonal), since $\cos 90^\circ = 0$. This gives a quick perpendicularity test used constantly in geometry and mechanics.

Example 5.6 — Dot product and angle between vectors

Find the angle between $\mathbf{a} = (1, 2, 2)$ and $\mathbf{b} = (2, -2, 1)$.

Solution

$\mathbf{a} \cdot \mathbf{b} = (1)(2) + (2)(-2) + (2)(1) = 2 - 4 + 2 = 0$.

$|\mathbf{a}| = \sqrt{(1+4+4)} = 3$, $|\mathbf{b}| = \sqrt{(4+4+1)} = 3$.

Since $\mathbf{a} \cdot \mathbf{b} = 0$ and both vectors are nonzero, $\cos\theta = 0$, so $\theta = 90^\circ$ — the vectors are perpendicular.

5.2.4 Cross Product of Two Vectors

The cross product (vector product) of $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$ produces a new vector, computed via the determinant:

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

which expands component-wise to:

$$\mathbf{a} \times \mathbf{b} = (a_2b_3 - a_3b_2, \quad a_3b_1 - a_1b_3, \quad a_1b_2 - a_2b_1)$$

Geometrical interpretation: the vector $\mathbf{a} \times \mathbf{b}$ is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$ (i.e., perpendicular to the plane containing them), with direction given by the right-hand rule, and magnitude:

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}| |\mathbf{b}| \sin\theta$$

which equals the area of the parallelogram spanned by $\mathbf{a}$ and $\mathbf{b}$. Consequently, if $\mathbf{a} \times \mathbf{b} = \mathbf{0}$ (with both vectors nonzero), $\mathbf{a}$ and $\mathbf{b}$ are parallel, since $\sin 0^\circ = 0$.

Cross product $\mathbf{a} \times \mathbf{b}$ (normal to plane of $\mathbf{a}$, $\mathbf{b}$)

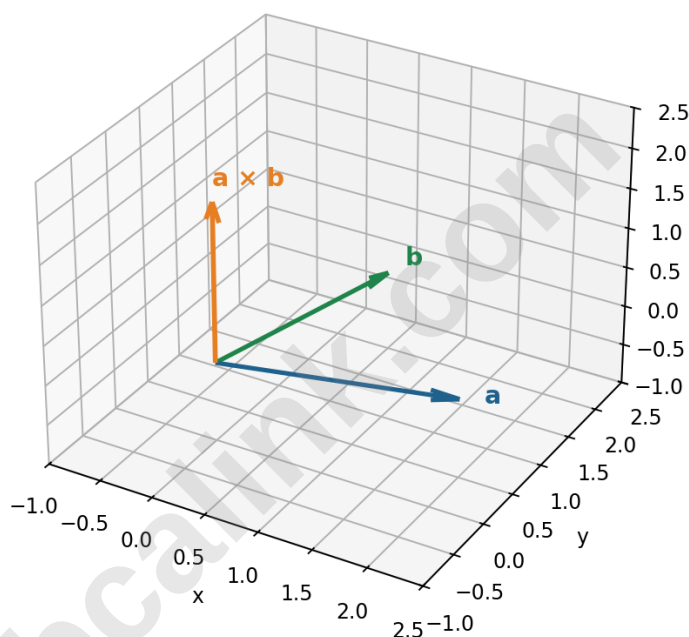


Figure 5.9 — Cross product $\mathbf{a} \times \mathbf{b}$ is normal to the plane of $\mathbf{a}$ and $\mathbf{b}$

Example 5.7 — Cross product and area of a parallelogram

Find $\mathbf{a} \times \mathbf{b}$ for $\mathbf{a} = (2, 0, 0)$ and $\mathbf{b} = (0, 3, 0)$, and interpret the result.

Solution

$$\mathbf{a} \times \mathbf{b} = (0 \cdot 0 - 0 \cdot 3, \quad 0 \cdot 0 - 2 \cdot 0, \quad 2 \cdot 3 - 0 \cdot 0) = (0, 0, 6).$$

$|\mathbf{a} \times \mathbf{b}| = 6$, which equals the area of the rectangle (a special parallelogram) with sides 2 and 3.

The result $(0, 0, 6)$ points along $+k$, perpendicular to both $\mathbf{a}$ and $\mathbf{b}$, as expected since $\mathbf{a}$ lies along x and $\mathbf{b}$ along y .

Note: Two vectors $\mathbf{u}$ and $\mathbf{v}$ are parallel exactly when $\mathbf{u} \times \mathbf{v} = \mathbf{0}$, and perpendicular exactly when $\mathbf{u} \cdot \mathbf{v} = 0$ — these two simple tests are used throughout mechanics and geometry to check orientation without computing angles directly.

Key Takeaways — Vectors in Space

- ✓ A vector in space is written $\mathbf{v} = (v_1, v_2, v_3)$ or $v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$; addition and scalar multiplication act component-wise.
- ✓ Length $|\mathbf{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2}$; distance between points uses the same formula on their difference vector.
- ✓ A unit vector has length 1 and is found via $\hat{\mathbf{v}} = \mathbf{v}/|\mathbf{v}|$; the null vector $\mathbf{0}$ has length 0.
- ✓ Dot product $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$ is a scalar; $\mathbf{a} \cdot \mathbf{b} = 0$ means $\mathbf{a} \perp \mathbf{b}$.
- ✓ Cross product $\mathbf{a} \times \mathbf{b}$ is a vector perpendicular to both $\mathbf{a}$ and $\mathbf{b}$, with $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}|\sin\theta = \text{area of the spanned parallelogram}$; $\mathbf{a} \times \mathbf{b} = \mathbf{0}$ means $\mathbf{a} \parallel \mathbf{b}$.

— End of Unit 5 —

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