

UNIT 4

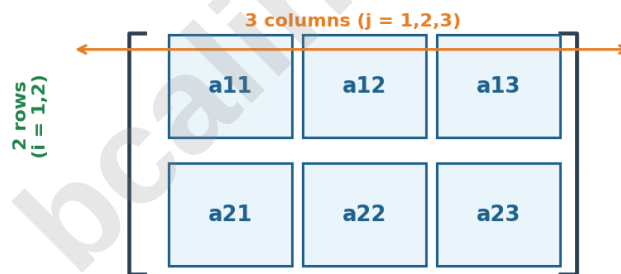
Matrices and Determinants

(8 Hours) — Mathematics I (CACS104)

Matrices provide a compact way to organize and manipulate large systems of numbers, and they are the natural language for describing linear equations, geometric transformations, and data. This unit builds the algebra of matrices from the ground up — types, operations, determinants, inverses — and connects matrices to linear and orthogonal transformations of the plane.

1. Introduction to Matrices

A matrix is a rectangular array of numbers (or other elements) arranged in rows and columns, enclosed in brackets. A matrix with m rows and n columns is said to have order (or size) $m \times n$, and is usually denoted by a capital letter such as A . The entry in the i -th row and j -th column is written a_{ij} .



A is a 2×3 matrix

Figure 4.1 — A 2×3 matrix: rows run horizontally, columns run vertically, and a_{ij} denotes the entry in row i , column j .

Example 4.1

Write the matrix A of order 2×2 whose entries are given by $a_{ij} = i + j$.

Solution

$$a_{11} = 1+1 = 2, a_{12} = 1+2 = 3, a_{21} = 2+1 = 3, a_{22} = 2+2 = 4$$

$$A = \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}$$

2. Types of Matrices

Type	Definition	Example
Row matrix	A matrix with a single row ($1 \times n$)	[1 2 3]
Column matrix	A matrix with a single column ($m \times 1$)	[[1],[2],[3]]
Square matrix	Number of rows equals number of columns ($n \times n$)	$2 \times 2, 3 \times 3, \dots$
Zero (Null) matrix	Every entry is 0	[[0,0],[0,0]]
Diagonal matrix	Square matrix; all off-diagonal entries are 0	[[3,0],[0,5]]
Scalar matrix	Diagonal matrix with all diagonal entries equal	[[4,0],[0,4]]
Identity matrix (I)	Scalar matrix with diagonal entries all 1	[[1,0],[0,1]]
Upper triangular	Square matrix; all entries below the diagonal are 0	[[1,2,3],[0,4,5],[0,0,6]]
Lower triangular	Square matrix; all entries above the diagonal are 0	[[1,0,0],[2,3,0],[4,5,6]]
Symmetric matrix	Square matrix with $A = A^T$ ($a_{ij} = a_{ji}$)	[[1,2],[2,5]]
Skew-symmetric matrix	Square matrix with $A^T = -A$ (so $a_{ii} = 0$)	[[0,3],[-3,0]]

3. Equality of Matrices

Two matrices A and B are said to be equal, written $A = B$, if and only if they have the same order (same number of rows and same number of columns) and every corresponding entry is equal: $a_{ij} = b_{ij}$ for all i, j .

Example 4.2

Find x and y if $[[x, 2], [3, y]] = [[4, 2], [3, 7]]$.

Solution

Comparing corresponding entries: $x = 4$ and $y = 7$.

4. Algebra of Matrices

4.1 Addition and Subtraction

Two matrices can be added or subtracted only if they have the same order. The sum (or difference) is obtained by adding (or subtracting) corresponding entries: $(A \pm B)_{ij} = a_{ij} \pm b_{ij}$.

4.2 Scalar Multiplication

For a scalar k and matrix A , kA is the matrix obtained by multiplying every entry of A by k :
 $(kA)_{ij} = k \cdot a_{ij}$.

4.3 Matrix Multiplication

Two matrices A (of order $m \times n$) and B (of order $n \times p$) can be multiplied only when the number of columns of A equals the number of rows of B . The product AB has order $m \times p$, and each entry is computed as a row-times-column sum:

Definition

$$(AB)_{ij} = \sum_k a_{ik} \cdot b_{kj} \text{ (sum over } k = 1 \text{ to } n)$$

Note: Matrix multiplication is generally NOT commutative — $AB \neq BA$ in general, and one product may not even be defined while the other is.

Example 4.3

If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix}$, find (a) $A + B$ (b) AB

Solution

(a) $A + B = \begin{bmatrix} 1+2 & 2+0 \\ 3+1 & 4+3 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 4 & 7 \end{bmatrix}$

(b) AB row 1: $(1 \cdot 2 + 2 \cdot 1, 1 \cdot 0 + 2 \cdot 3) = (4, 6)$

AB row 2: $(3 \cdot 2 + 4 \cdot 1, 3 \cdot 0 + 4 \cdot 3) = (10, 12)$

So $AB = \begin{bmatrix} 4 & 6 \\ 10 & 12 \end{bmatrix}$

5. Determinant of a Matrix

The determinant of a square matrix A , written $\det(A)$ or $|A|$, is a single number computed from its entries that captures important information about the matrix — including whether it has an inverse.

Order	Matrix	Determinant formula
2×2	$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$	$ A = ad - bc$
3×3	$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$	$ A = a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2)$

Example 4.4

Evaluate the determinant of $A = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}$.

Solution

$$|A| = (2)(1) - (3)(4) = 2 - 12 = -10$$

6. Transpose of a Matrix

The transpose of a matrix A , written A^T , is obtained by interchanging its rows and columns: if A has order $m \times n$, then A^T has order $n \times m$, and $(A^T)_{ij} = a_{ji}$.

Transpose: rows of A become columns of A^T

A (2×3)

1	2	3
4	5	6

A^T (3×2)

1	4
2	5
3	6

Figure 4.2 — Taking the transpose of a 2×3 matrix gives a 3×2 matrix; row 1 of A becomes column 1 of A^T .

Properties of the transpose, for matrices A , B and scalar k :

- $(A^T)^T = A$
- $(A + B)^T = A^T + B^T$
- $(kA)^T = kA^T$
- $(AB)^T = B^T A^T$ (note the reversed order)

7. Minors and Cofactors of a Matrix

For a square matrix A , the minor M_{ij} of the entry a_{ij} is the determinant of the smaller matrix obtained by deleting row i and column j from A . The cofactor C_{ij} attaches a sign to the minor:

Definition

$$C_{ij} = (-1)^{(i+j)} \cdot M_{ij}$$

Cofactors are the building blocks used both to expand a determinant along any row or column, and to construct the adjoint of a matrix.

Example 4.5

For $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 10 \end{bmatrix}$, find the minor and cofactor of the entry $a_{12} = 2$.

Solution

Delete row 1 and column 2: remaining matrix = $\begin{bmatrix} 4 & 6 \\ 7 & 10 \end{bmatrix}$

$$M_{12} = (4)(10) - (6)(7) = 40 - 42 = -2$$

$$C_{12} = (-1)^{(1+2)} \cdot M_{12} = (-1)(-2) = 2$$

8. Properties of Determinants

The following properties (stated without proof, as per the syllabus) hold for square matrices and are frequently used to simplify determinant calculations before expanding:

- If two rows (or columns) of a matrix are identical, its determinant is 0.
- Interchanging any two rows (or columns) changes the sign of the determinant.
- If every entry of one row (or column) is multiplied by k , the determinant is multiplied by k .
- The determinant of a matrix equals the determinant of its transpose: $|A| = |A^T|$.
- If a matrix has a full row or column of zeros, its determinant is 0.
- Adding a multiple of one row (or column) to another does not change the determinant.
- For square matrices A and B of the same order, $|AB| = |A| \cdot |B|$.
- The determinant of a triangular (or diagonal) matrix is the product of its diagonal entries.

9. Singular and Non-Singular Matrix

Term	Condition	Consequence
Singular matrix	$ A = 0$	A has no inverse
Non-singular matrix	$ A \neq 0$	A has a unique inverse A^{-1}

Example 4.6

Is $A = \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}$ singular or non-singular?

Solution

$$|A| = (2)(2) - (4)(1) = 4 - 4 = 0$$

Since $|A| = 0$, A is a singular matrix (it has no inverse).

10. Adjoint and Inverse of a Matrix

The adjoint (or adjugate) of a square matrix A , written $\text{adj}(A)$, is the transpose of the matrix of cofactors of A :

Definition

$\text{adj}(A) = [C_{ij}]^T$ (transpose of the cofactor matrix)

When A is non-singular ($|A| \neq 0$), its inverse is given by:

Key Formula

$$A^{-1} = \text{adj}(A) / |A|$$

The inverse satisfies $A \cdot A^{-1} = A^{-1} \cdot A = I$, the identity matrix — the matrix analogue of the reciprocal of a non-zero number.

Example 4.7

Find the inverse of $A = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}$.

Solution

$$|A| = (2)(3) - (1)(5) = 6 - 5 = 1 \text{ (non-singular, since } |A| \neq 0 \text{)}$$

$$\text{For a } 2 \times 2 \text{ matrix } \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \text{adj}(A) = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}$$

$$A^{-1} = \text{adj}(A)/|A| = \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix} / 1 = \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}$$

11. Linear Transformations

A linear transformation T from one vector space to another is a function that preserves vector addition and scalar multiplication:

Definition

$$T(u + v) = T(u) + T(v) \text{ and } T(k \cdot u) = k \cdot T(u)$$

Every linear transformation on $\mathbb{R}^n$ can be represented by multiplication with a matrix: $T(x) = Ax$ for some matrix A . Applying T to the standard unit vectors $e_1, e_2, \dots$ gives exactly the columns of A .

A Linear Transformation Maps the Unit Square to a Parallelogram

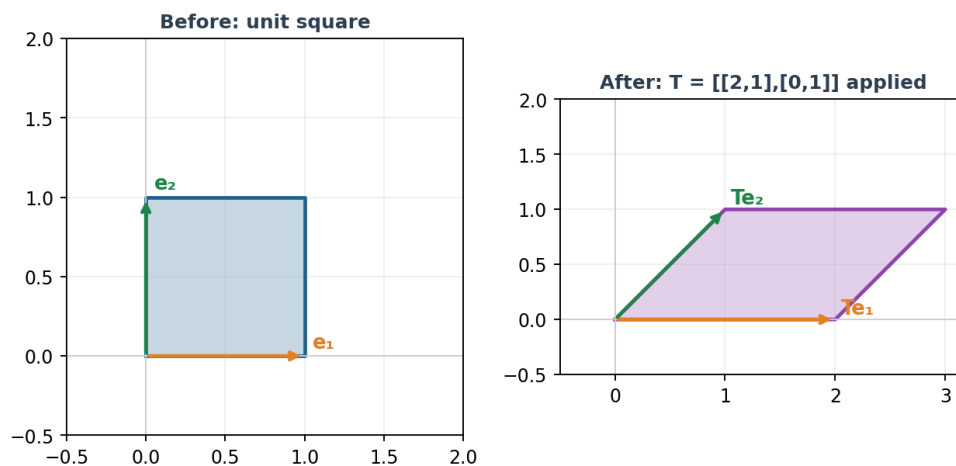


Figure 4.3 — The matrix $T = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$ transforms the unit square into a parallelogram; straight lines and the origin are preserved, as required of any linear transformation.

Example 4.8

If $T(x, y) = (2x + y, x - y)$, find $T(1, 2)$ and write T as a matrix.

Solution

$$T(1, 2) = (2(1) + 2, 1 - 2) = (4, -1)$$

$$\text{Matrix form: } T([x, y]) = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix}$$

12. Orthogonal Transformations

A linear transformation represented by a matrix R is called orthogonal if it preserves lengths and angles — equivalently, if $R^T \cdot R = I$, meaning $R^T = R^{-1}$. Rotations and reflections of the plane are the classic examples of orthogonal transformations.

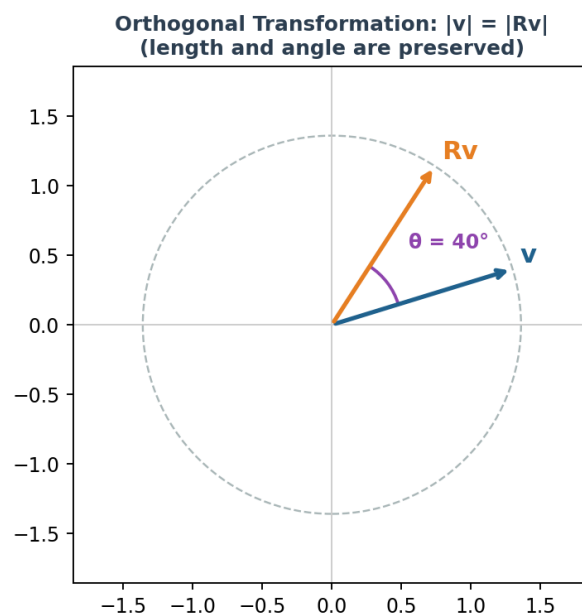


Figure 4.4 — A rotation matrix R applied to vector v produces Rv with the same length: $|v| = |Rv|$. Distances and angles between vectors are unchanged.

A 2-D rotation by angle θ is represented by the matrix:

Rotation Matrix

$$R(\theta) = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

Note: Since a rotation matrix satisfies $R^T R = I$, its determinant is always ± 1 — rotations preserve area as well as length.

13. Rank of a Matrix

The rank of a matrix A , written $\text{rank}(A)$ or $\rho(A)$, is the maximum number of linearly independent rows (equivalently, columns) in A . In practice, rank is found by reducing A to row-echelon form using elementary row operations and counting the number of non-zero rows.

1	2	-1
0	1	3
0	0	0

Row-echelon form: 2 non-zero (pivot) rows $\Rightarrow$ rank = 2

Figure 4.5 — After row reduction, the number of non-zero (pivot) rows gives the rank of the matrix.

For an $m \times n$ matrix, the rank can be at most $\min(m, n)$. A square matrix of order n has full rank n if and only if it is non-singular ($|A| \neq 0$).

Example 4.9

Find the rank of $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 1 & 1 \end{bmatrix}$.

Solution

$R_2 \rightarrow R_2 - 2R_1: \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$

$R_3 \rightarrow R_3 - R_1: \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & -1 & -2 \end{bmatrix}$

Swap R_2 and $R_3: \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$

Two non-zero rows remain, so $\text{rank}(A) = 2$.

Key Takeaways

- ✓ A matrix of order $m \times n$ is a rectangular array; special types (diagonal, identity, symmetric, triangular, etc.) have distinguishing structural patterns.
- ✓ Two matrices are equal only if they share the same order and every corresponding entry matches.
- ✓ Addition/subtraction require matching order; multiplication AB requires columns of A to match rows of B , and is generally not commutative.
- ✓ $\det(A)$ is computed from cofactor expansion; a matrix is singular if $|A| = 0$ and non-singular if $|A| \neq 0$.
- ✓ The transpose A^T swaps rows and columns; minors and cofactors are the building blocks of both determinants and the adjoint.
- ✓ $A^{-1} = \text{adj}(A)/|A|$ exists only when A is non-singular, and satisfies $A \cdot A^{-1} = I$.
- ✓ Linear transformations $T(x) = Ax$ preserve vector addition and scalar multiplication; orthogonal transformations additionally preserve length and angle ($R^T R = I$).
- ✓ The rank of a matrix is the number of non-zero rows after row reduction to echelon form — a measure of linear independence among rows/columns.

— End of Unit 4 —