

UNIT 2

Two Dimensional and Three Dimensional Transformation

(7 Hours) — Computer Graphics and Animation (CACS305)

2.1 Introduction to 2-Dimensional Transformation

A geometric transformation is an operation that changes the position, size, orientation, or shape of an object defined in a coordinate system. In computer graphics, every object — a point, a line, a polygon, or a complex model — is described using coordinates, and transformations are used to manipulate these coordinates mathematically so that the object can be moved, resized, rotated, or reshaped on the screen.

A 2D transformation takes a point (x, y) and maps it to a new point (x', y') using a transformation rule. The four basic transformations studied in this unit are:

- Translation — shifting an object from one position to another
- Rotation — turning an object about a fixed reference point
- Scaling — changing the size of an object
- Reflection and Shear — mirroring and slanting an object

Note: Transformations that preserve distances and angles (translation, rotation, reflection) are called rigid-body transformations. Scaling and shear are non-rigid because they change the shape or size of the object.

2.2 2-D Translation, Rotation, Scaling

2.2.1 Translation

Translation repositions an object along a straight-line path from one location to another without changing its shape, size, or orientation. Every point of the object is displaced by the same amount.

If a point $P(x, y)$ is translated by a distance t_x along the X-axis and t_y along the Y-axis, the new coordinates (x', y') are given by:

$$x' = x + t_x \quad y' = y + t_y$$

Here, (t_x, t_y) is called the translation vector or shift vector. In matrix form:

$$[x' \ y'] = [x \ y] + [t_x \ t_y]$$

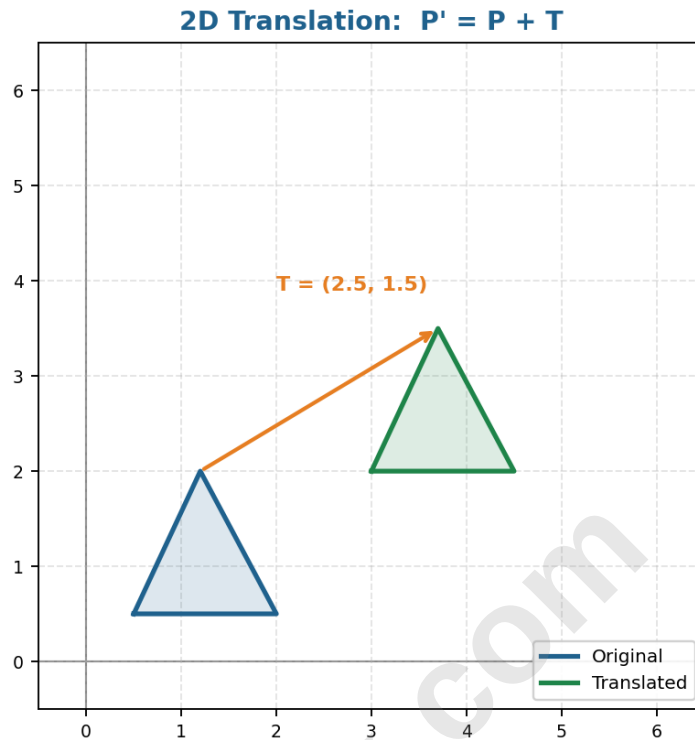


Fig 2.1 — A triangle translated by vector $T = (2.5, 1.5)$

Example 2.1

A point $P(2, 3)$ is translated by $t_x = 5$, $t_y = 4$. Find the new position P' .

Solution

$$x' = x + t_x = 2 + 5 = 7$$

$$y' = y + t_y = 3 + 4 = 7$$

Therefore, $P'(7, 7)$.

2.2.2 Rotation

Rotation moves an object along a circular path about a fixed point called the pivot point or origin of rotation, through a specified angle θ . A positive angle produces a counter-clockwise rotation, while a negative angle produces a clockwise rotation (using the standard mathematical convention).

For a point $P(x, y)$ rotated by angle θ about the origin, the new coordinates are derived using trigonometric identities:

$$x' = x \cdot \cos \theta - y \cdot \sin \theta \quad y' = x \cdot \sin \theta + y \cdot \cos \theta$$

In matrix form, this is written as:

$$[x' \ y'] = [x \ y] \cdot R(\theta), \quad \text{where } R(\theta) = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}$$

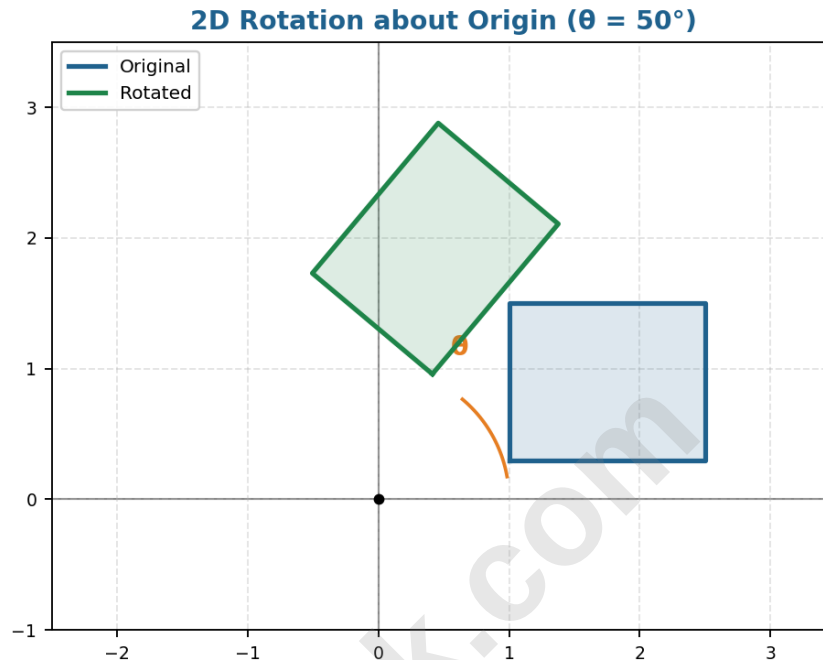


Fig 2.2 — A square rotated 50° counter-clockwise about the origin

Example 2.2

Rotate the point $P(1, 0)$ by $\theta = 90^\circ$ about the origin.

Solution

$$x' = 1 \cdot \cos 90^\circ - 0 \cdot \sin 90^\circ = 1(0) - 0(1) = 0$$

$$y' = 1 \cdot \sin 90^\circ + 0 \cdot \cos 90^\circ = 1(1) + 0(0) = 1$$

Therefore, $P'(0, 1)$.

2.2.3 Scaling

Scaling changes the size of an object. Each coordinate of every point is multiplied by a scaling factor: S_x along the X-axis and S_y along the Y-axis.

$$x' = x \cdot S_x \quad y' = y \cdot S_y$$

If $S_x = S_y$, the scaling is uniform and the shape of the object is preserved. If $S_x \neq S_y$, the scaling is differential and the object becomes distorted. Values greater than 1 enlarge the object; values between 0 and 1 shrink it.

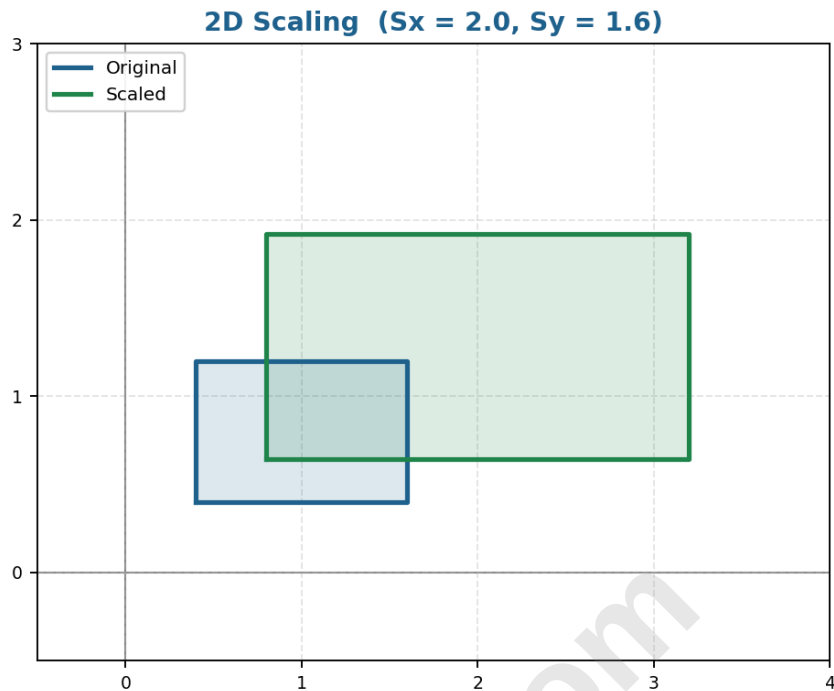


Fig 2.3 — A rectangle scaled with $S_x = 2.0$, $S_y = 1.6$

Example 2.3

A rectangle with a corner at $P(2, 3)$ is scaled with $S_x = 3$, $S_y = 2$. Find P' .

Solution

$$x' = x \cdot S_x = 2 \times 3 = 6$$

$$y' = y \cdot S_y = 3 \times 2 = 6$$

Therefore, $P'(6, 6)$.

Note: Scaling with respect to the origin also changes the object's position, not just its size — unless the object is centered exactly at the origin. To scale about an arbitrary fixed point, the object must first be translated so that the fixed point coincides with the origin, then scaled, then translated back.

2.3 Homogeneous Coordinates, Reflection, Shear Transform

2.3.1 Homogeneous Coordinates

A major limitation of the equations above is that translation is expressed as addition while rotation and scaling are expressed as multiplication. This makes it impossible to combine (compose) multiple transformations into a single matrix using simple matrix multiplication.

To solve this, we use homogeneous coordinates. Every 2D point (x, y) is represented as a 3-element vector $(x, y, 1)$ by adding an extra coordinate, $h = 1$. This allows all transformations

— translation, rotation, scaling, reflection, and shear — to be expressed uniformly as 3×3 matrices, so that a sequence of transformations can be combined by simply multiplying their matrices together.

Transformation	Homogeneous Matrix (3×3)
Translation	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ t_x & t_y & 1 \end{bmatrix}$
Rotation (θ)	$\begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$
Scaling	$\begin{bmatrix} S_x & 0 & 0 \\ 0 & S_y & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Using homogeneous coordinates, a point transformed by translation followed by rotation, for example, can be computed as:

$$[x' \ y' \ 1] = [x \ y \ 1] \cdot T \cdot R$$

Note: The order of matrix multiplication matters — transformation composition is not commutative. Rotating and then translating generally produces a different result than translating and then rotating.

2.3.2 Reflection

Reflection produces a mirror image of an object about a chosen reflection axis (a line acting like a mirror). The size of the object does not change, but its orientation is flipped.

Reflection About	Transformation Equations	Matrix
X-axis	$x' = x, y' = -y$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
Y-axis	$x' = -x, y' = y$	$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
Origin	$x' = -x, y' = -y$	$\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
Line $y = x$	$x' = y, y' = x$	$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

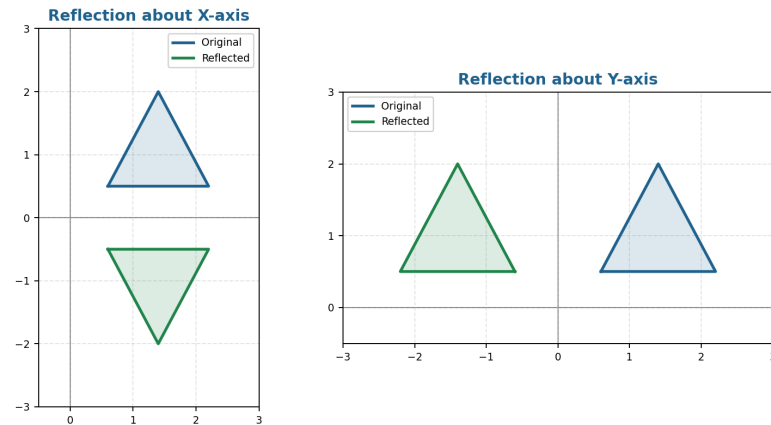


Fig 2.4 — Reflection of a triangle about the X-axis and Y-axis

Example 2.4

Reflect the point P(4, 5) about (a) the X-axis, and (b) the Y-axis.

Solution

(a) About X-axis: $x' = x = 4$, $y' = -y = -5 \rightarrow P'(4, -5)$

(b) About Y-axis: $x' = -x = -4$, $y' = y = 5 \rightarrow P'(-4, 5)$

2.3.3 Shear Transform

Shear is a transformation that slants (distorts) the shape of an object along the X-axis, the Y-axis, or both. It shifts each point by an amount proportional to its distance from a reference line.

$$\text{X-Shear: } x' = x + Sh_x \cdot y, \quad y' = y$$

$$\text{Y-Shear: } x' = x, \quad y' = y + Sh_y \cdot x$$

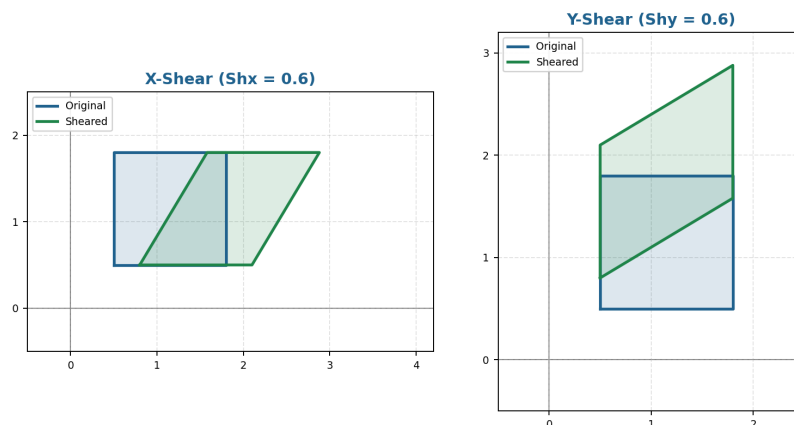


Fig 2.5 — X-shear and Y-shear applied to a square ($Sh = 0.6$)

Example 2.5

Apply an X-shear of $Sh_x = 2$ to the point $P(3, 4)$.

Solution

$$x' = x + Sh_x \cdot y = 3 + 2(4) = 11$$

$$y' = y = 4$$

Therefore, $P'(11, 4)$.

Key Takeaways — 2-D Transformations

- ✓ Translation shifts, rotation turns, scaling resizes, and shear slants an object.
- ✓ Homogeneous coordinates $(x, y, 1)$ allow all transformations to be represented as 3×3 matrices.
- ✓ Homogeneous coordinates make it possible to combine multiple transformations by matrix multiplication.
- ✓ Reflection creates a mirror image about a chosen axis or line without changing size.
- ✓ Transformation order matters — matrix multiplication is not commutative.

2.4 3-Dimensional Transformation

Three-dimensional transformations extend the same principles used in 2D to objects defined in 3D space using coordinates (x, y, z). Since real-world objects and scenes are three-dimensional, 3D transformations are essential in computer graphics, animation, CAD, and virtual reality applications for positioning, orienting, and resizing objects within a 3D scene.

Just as in 2D, homogeneous coordinates are used for 3D transformations. A point (x, y, z) is represented as a 4-element vector (x, y, z, 1), and every transformation is expressed as a 4×4 matrix. This allows translation, rotation, scaling, reflection, and shear in 3D to be composed together by matrix multiplication, exactly as in the 2D case.

The general homogeneous coordinate transformation in 3D takes the form:

$$[x' \ y' \ z' \ 1] = [x \ y \ z \ 1] \cdot M_{4 \times 4}$$

3D rotation is more complex than 2D rotation because rotation can occur about any of the three coordinate axes — X, Y, or Z — each producing a distinct rotation matrix, unlike 2D where rotation is always about a single point (the origin or a pivot).

Note: In 3D graphics, rotations about the X, Y, and Z axes are often called roll, pitch, and yaw respectively in aviation and robotics contexts.

2.5 3-D Translation, Rotation, Scaling, Reflection, Shear

2.5.1 3-D Translation

A point P(x, y, z) is translated to P'(x', y', z') by adding translation distances t_x , t_y , t_z along the respective axes:

$$x' = x + t_x, \quad y' = y + t_y, \quad z' = z + t_z$$

$$\text{Matrix } T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ t_x & t_y & t_z & 1 \end{bmatrix}$$

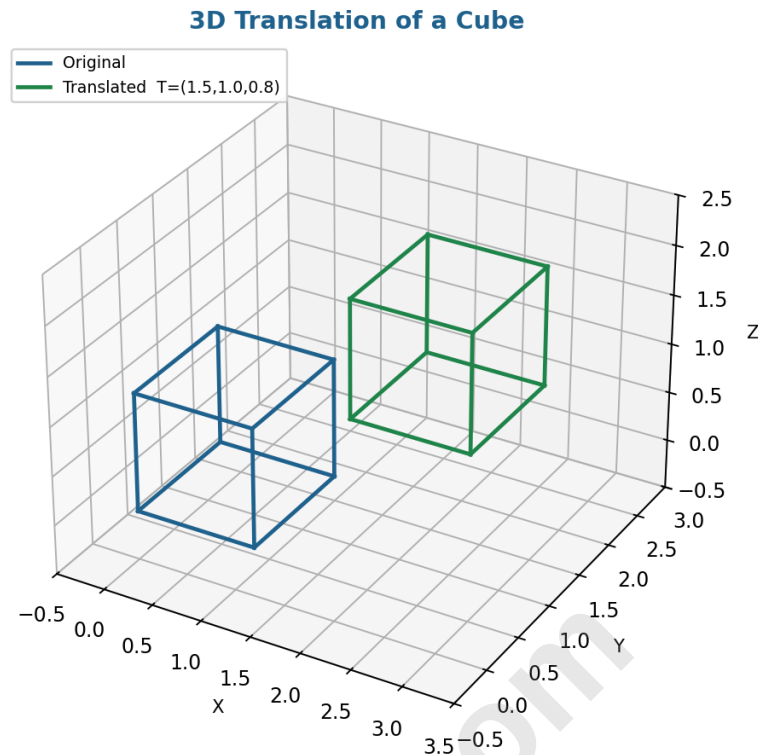


Fig 2.6 — A cube translated by vector $T = (1.5, 1.0, 0.8)$

Example 2.6

Translate the point $P(1, 2, 3)$ by $t_x = 2$, $t_y = -1$, $t_z = 4$.

Solution

$$x' = 1 + 2 = 3, y' = 2 + (-1) = 1, z' = 3 + 4 = 7$$

Therefore, $P'(3, 1, 7)$.

2.5.2 3-D Rotation

3D rotation turns an object about one of the coordinate axes by an angle θ . Each axis has its own rotation matrix:

Axis	Equations	Matrix
X-axis	$y' = y \cdot \cos\theta - z \cdot \sin\theta$ $z' = y \cdot \sin\theta + z \cdot \cos\theta$ $x' = x$	$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
Y-axis	$z' = z \cdot \cos\theta - x \cdot \sin\theta$ $x' = z \cdot \sin\theta + x \cdot \cos\theta$ $y' = y$	$\begin{bmatrix} \cos\theta & 0 & -\sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ \sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Axis	Equations	Matrix
Z-axis	$x' = x \cdot \cos\theta - y \cdot \sin\theta$ $y' = x \cdot \sin\theta + y \cdot \cos\theta$ $z' = z$	$\begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

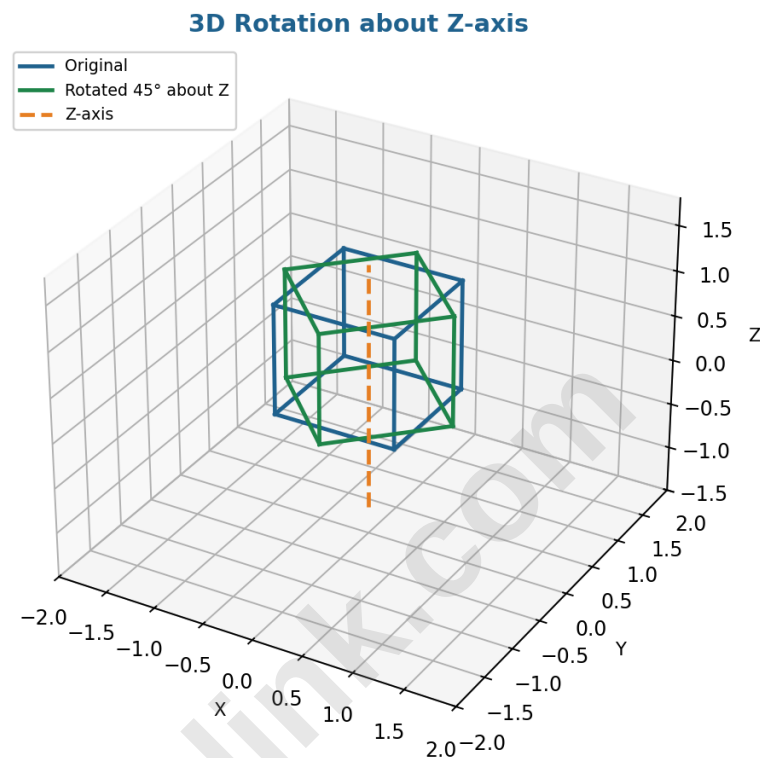


Fig 2.7 — A cube rotated 45° about the Z-axis

Example 2.7

Rotate the point $P(1, 0, 0)$ by $\theta = 90^\circ$ about the Z-axis.

Solution

$$x' = x \cdot \cos 90^\circ - y \cdot \sin 90^\circ = 1(0) - 0(1) = 0$$

$$y' = x \cdot \sin 90^\circ + y \cdot \cos 90^\circ = 1(1) + 0(0) = 1$$

$$z' = z = 0$$

Therefore, $P'(0, 1, 0)$.

2.5.3 3-D Scaling

Scaling in 3D resizes an object along each axis independently using scale factors S_x, S_y, S_z :

$$x' = x \cdot S_x, \quad y' = y \cdot S_y, \quad z' = z \cdot S_z$$

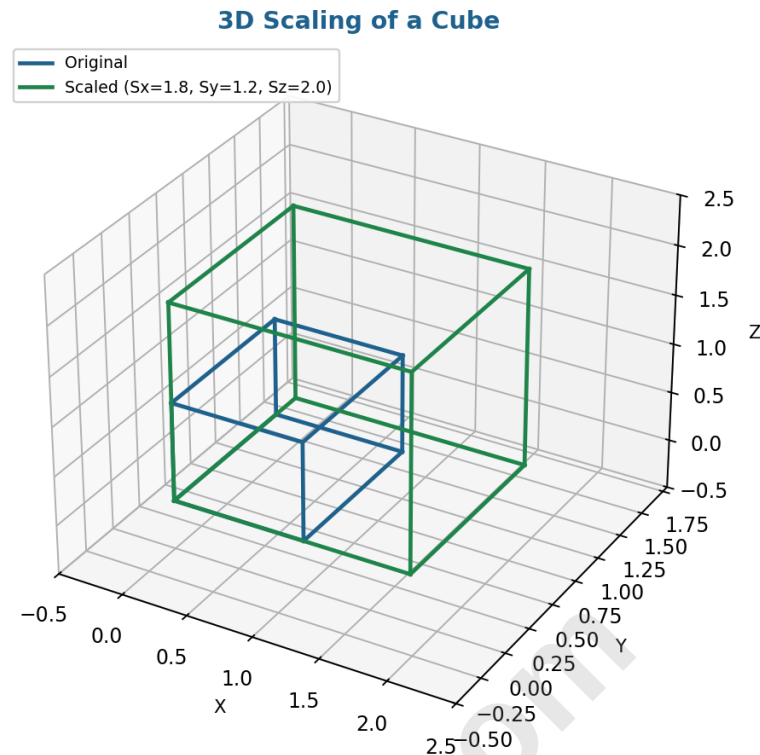


Fig 2.8 — A cube scaled with $S_x = 1.8$, $S_y = 1.2$, $S_z = 2.0$

Example 2.8

Scale the point $P(2, 3, 4)$ with $S_x = 2$, $S_y = 1$, $S_z = 3$.

Solution

$$x' = 2 \times 2 = 4, y' = 3 \times 1 = 3, z' = 4 \times 3 = 12$$

Therefore, $P'(4, 3, 12)$.

2.5.4 3-D Reflection

Reflection in 3D mirrors an object about a chosen coordinate plane (XY, YZ, or ZX).

Reflection About	Effect	Matrix (diagonal)
XY-plane	z coordinate is negated	$\text{diag}(1, 1, -1, 1)$
YZ-plane	x coordinate is negated	$\text{diag}(-1, 1, 1, 1)$
ZX-plane	y coordinate is negated	$\text{diag}(1, -1, 1, 1)$

Example 2.9

Reflect the point $P(3, 4, 5)$ about the XY-plane.

Solution

Reflection about XY-plane negates z: $x' = 3$, $y' = 4$, $z' = -5$

Therefore, $P'(3, 4, -5)$.

2.5.5 3-D Shear

Shear in 3D slants an object along one axis in proportion to the coordinate values of the other two axes. For example, a Z-shear (shearing in the direction of X and Y with respect to Z) is given by:

$$x' = x + Sh_x \cdot z, \quad y' = y + Sh_y \cdot z, \quad z' = z$$

Similarly, X-shear and Y-shear displace coordinates in proportion to x or y respectively, following the same pattern as the 2D shear equations extended into the third dimension.

Example 2.10

Apply a Z-shear with $Sh_x = 1$, $Sh_y = 2$ to the point $P(2, 3, 4)$.

Solution

$$x' = x + Sh_x \cdot z = 2 + 1(4) = 6$$

$$y' = y + Sh_y \cdot z = 3 + 2(4) = 11$$

$$z' = z = 4$$

Therefore, $P'(6, 11, 4)$.

Key Takeaways — 3-D Transformations

- ✓ 3D points use 4-element homogeneous coordinates $(x, y, z, 1)$ and 4×4 transformation matrices.
- ✓ Rotation in 3D requires a choice of axis — X, Y, or Z — each with its own matrix.
- ✓ Translation, scaling, and shear extend directly from their 2D counterparts to three axes.
- ✓ Reflection in 3D mirrors an object about a coordinate plane rather than an axis.
- ✓ Combined 3D transformations are computed by multiplying the individual 4×4 matrices in sequence.

— End of Unit 2 —