

UNIT 6

Solution of Partial Differential Equations

(5 Hours) — Numerical Methods (CACS252)

6.1 Introduction to Partial Differential Equations

A **partial differential equation (PDE)** involves an unknown function of two or more independent variables, e.g. $u(x, y)$, and its partial derivatives. The **order** is the highest partial derivative present. PDEs describe heat flow, vibrations, fluid flow, electrostatics and many other physical fields.

A general linear second-order PDE in two variables is

$$A \frac{\partial^2 u}{\partial x^2} + B \frac{\partial^2 u}{\partial x \partial y} + C \frac{\partial^2 u}{\partial y^2} + (\text{lower-order terms}) = 0$$

and it is classified by the sign of the discriminant $B^2 - 4AC$:

Discriminant	Type	Model equation	Physical meaning
$B^2 - 4AC < 0$	Elliptic	Laplace: $u_{xx} + u_{yy} = 0$; Poisson: $u_{xx} + u_{yy} = f(x, y)$	Steady-state (equilibrium) problems
$B^2 - 4AC = 0$	Parabolic	Heat: $u_t = c^2 u_{xx}$	Diffusion, time-dependent heat flow
$B^2 - 4AC > 0$	Hyperbolic	Wave: $u_{tt} = c^2 u_{xx}$	Vibrations and waves

This unit treats the elliptic equations of Laplace and Poisson. They are **boundary value problems**: u is specified on the boundary of a region R (Dirichlet condition), and we must find u inside. Since exact solutions are available only for very simple regions, we replace derivatives by differences and solve the resulting algebraic system.

6.2 Deriving Difference Equations

Cover the region with a rectangular grid of points $(x_i, y_j) = (ih, jk)$, where h and k are the step sizes in x and y . Write $u_{i,j} \approx u(x_i, y_j)$. Adding the Taylor series $u(x \pm h, y) = u \pm h u_x + (h^2/2) u_{xx} \pm (h^3/6) u_{xxx} + (h^4/24) u_{xxxx} \pm \dots$ gives the **central difference approximations**:

$$u_x \approx (u_{i+1,j} - u_{i-1,j}) / (2h) \quad u_y \approx (u_{i,j+1} - u_{i,j-1}) / (2k) \quad (\text{error } O(h^2))$$

$$u_{xx} \approx (u_{i+1,j} - 2u_{i,j} + u_{i-1,j}) / h^2 \quad u_{yy} \approx (u_{i,j+1} - 2u_{i,j} + u_{i,j-1}) / k^2 \quad (\text{error } O(h^2))$$

6.2.1 Laplace's Equation $\nabla^2 u = u_{xx} + u_{yy} = 0$

Substituting into $u_{xx} + u_{yy} = 0$ with square mesh $h = k$ gives

$$u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j} = 0, \text{ i.e. } u_{i,j} = \frac{1}{4} (u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1})$$

This is the **standard five-point formula**: the value at a point is the **average of its four neighbours** (a discrete form of the mean-value property of harmonic functions). If the axes are rotated by 45° we get the **diagonal five-point formula** $u_{i,j} = \frac{1}{4} (u_{i+1,j+1} + u_{i-1,j+1} + u_{i+1,j-1} + u_{i-1,j-1})$, which is used to obtain a good starting guess.

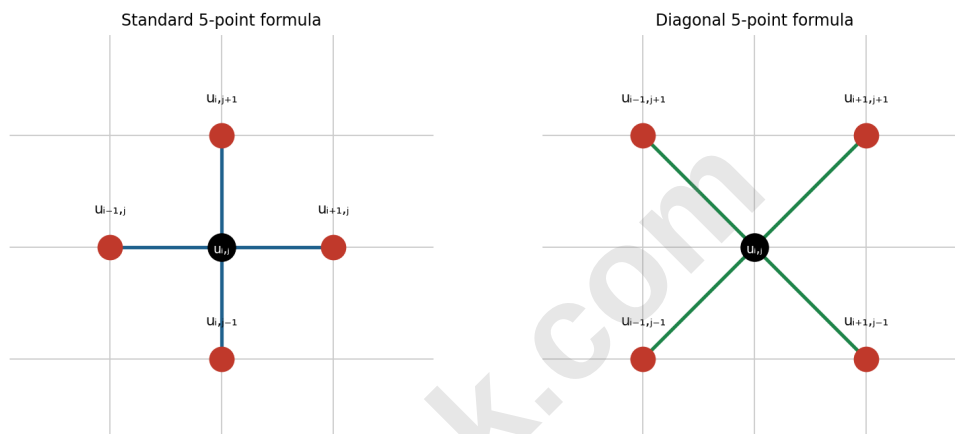


Figure 6.1 — Standard (left) and diagonal (right) five-point stencils for Laplace's equation.

6.2.2 Poisson's Equation $\nabla^2 u = f(x, y)$

With $h = k$ the same substitution gives

$$u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j} = h^2 f(x_i, y_j)$$

and for the diagonal stencil: $u_{i+1,j+1} + u_{i-1,j+1} + u_{i+1,j-1} + u_{i-1,j-1} - 4u_{i,j} = 2h^2 f(x_i, y_j)$. Poisson's equation reduces to Laplace's when $f = 0$. If $h \neq k$, use u_{xx} and u_{yy} with their own steps: $(u_{i+1,j} - 2u_{i,j} + u_{i-1,j})/h^2 + (u_{i,j+1} - 2u_{i,j} + u_{i,j-1})/k^2 = f$.

Note: Each interior grid point gives one linear equation, so for N interior points we get N equations in N unknowns. The coefficient matrix (4 on the diagonal, -1 for each neighbour) is sparse, symmetric and diagonally dominant, so iterative methods converge.

6.3 Solving the Difference Equations

6.3.1 Direct solution

For a small grid write the equations for all interior points and solve the linear system by Gaussian elimination (Unit 4). Symmetry of the region and boundary values can reduce the number of unknowns.

Example 6.1

Solve Laplace's equation on a square with 2×2 interior points u_1, u_2 (top row) and u_3, u_4 (bottom row). The top edge is held at $u = 100$ and the other three edges at $u = 0$.

Solution

Each interior value is the average of its four neighbours:

$$u_1 = \frac{1}{4}(100 + 0 + u_2 + u_3) \Rightarrow 4u_1 - u_2 - u_3 = 100$$

$$u_2 = \frac{1}{4}(100 + 0 + u_1 + u_4) \Rightarrow 4u_2 - u_1 - u_4 = 100$$

$$u_3 = \frac{1}{4}(u_1 + 0 + 0 + u_4) \Rightarrow 4u_3 - u_1 - u_4 = 0$$

$$u_4 = \frac{1}{4}(u_2 + 0 + 0 + u_3) \Rightarrow 4u_4 - u_2 - u_3 = 0$$

By the left-right symmetry $u_1 = u_2 = a$ and $u_3 = u_4 = b$, so $3a - b = 100$ and $3b - a = 0$. Then $a = 3b$, $8b = 100$:

$$u_1 = u_2 = 37.5, u_3 = u_4 = 12.5$$

6.3.2 Iterative solution — Liebmann's method

For larger grids the systems are solved by Gauss-Seidel iteration, called **Liebmann's method** in this context: sweep over the interior points (say left to right, top to bottom) and replace each value by the average of its four neighbours, using the newest values available. Repeat until the largest change is smaller than the tolerance. Convergence can be accelerated by **successive over-relaxation (SOR)**: $u_{i,j}(\text{new}) = (1 - \omega) u_{i,j}(\text{old}) + \omega \times (\text{average of neighbours})$, with $1 < \omega < 2$.

Algorithm

- Set the boundary values. Choose initial values for interior points (0, or the diagonal-formula estimate, or the mean of the boundary).
- For each interior point in a fixed order compute $u_{i,j} = \frac{1}{4}(u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - h^2 f_{i,j})$ (with $f = 0$ for Laplace).
- If $\max |\text{new} - \text{old}| < \epsilon$ for all points, stop; otherwise repeat step 2.

Example 6.2

A square plate has its top edge at 100° and the other three edges at 0° . Using a 3×3 array of interior points $u_1 \dots u_9$ (numbered row by row from the top-left), find the steady-state temperatures by Liebmann's method starting from $u = 0$.

Setting up

Equations (rows from the top): $4u_1 - u_2 - u_4 = 100$, $4u_2 - u_1 - u_3 - u_5 = 100$, $4u_3 - u_2 - u_6 = 100$, $4u_4 - u_1 - u_5 - u_7 = 0$, $4u_5 - u_2 - u_4 - u_6 - u_8 = 0$, $4u_6 - u_3 - u_5 - u_9 = 0$, $4u_7 - u_4 - u_8 = 0$, $4u_8 - u_5 - u_7 - u_9 = 0$, $4u_9 - u_6 - u_8 = 0$.

Liebmann updates in order $u_1, u_2, \dots, u_9$, e.g. first sweep: $u_1 = (100 + 0)/4 = 25$, $u_2 = (100 + 25)/4 = 31.25$, $u_3 = (100 + 31.25)/4 = 32.8125$, ...

Sweep	u_1	u_2	u_3	u_4	u_5	u_6	u_7	u_8	u_9
0	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
1	25.00	31.25	32.81	6.25	9.38	10.55	1.56	2.73	3.32
2	34.38	44.14	38.67	11.33	17.19	14.79	3.52	6.01	5.20
3	38.87	48.68	40.87	14.89	21.09	16.79	5.22	7.88	6.17
5	41.88	51.70	42.37	17.77	24.02	18.26	6.66	9.33	6.90
10	42.83	52.65	42.84	18.72	24.97	18.73	7.13	9.81	7.14
Exact	42.86	52.68	42.86	18.75	25.00	18.75	7.14	9.82	7.14

All nine values are within 0.001 of the exact solution of the linear system after 15 sweeps. As a check, the centre value is exactly $25 = 100/4$: rotating the plate by 90° three times and adding the four solutions gives $u = 100$ at every point, and by symmetry the centre receives the same amount from each of the four edges.

Laplace solution on a 5×5 grid (top edge = 100, other edges = 0)

—	100	100	100	—
0	42.86	52.68	42.86	0
0	18.75	25.00	18.75	0
0	7.14	9.82	7.14	0
—	0	0	0	—

Figure 6.2 — Steady-state temperatures on the 5×5 grid (boundary values in bold). Values decrease away from the hot top edge.

6.3.3 Poisson's equation

Example 6.3

Solve $\nabla^2 u = -10(x^2 + y^2 + 10)$ on the square $0 \leq x, y \leq 3$ with $u = 0$ on the boundary, using $h = 1$. The interior points are $(1, 1)$, $(2, 1)$, $(1, 2)$, $(2, 2)$; call them u_1 , u_2 , u_3 , u_4 .

Solution

Poisson formula with $h = 1$: (sum of four neighbours) $- 4u = f(x, y)$. The values of f are:
 $f(1,1) = -10(1 + 1 + 10) = -120$, $f(2,1) = f(1,2) = -10(4 + 1 + 10) = -150$, $f(2,2) = -10(4 + 4 + 10) = -180$

Equations (boundary neighbours are 0):

$$u_2 + u_3 - 4u_1 = -120$$

$$u_1 + u_4 - 4u_2 = -150$$

$$u_1 + u_4 - 4u_3 = -150$$

$$u_2 + u_3 - 4u_4 = -180$$

By symmetry $u_2 = u_3$, so $2u_2 - 4u_1 = -120$ and $2u_2 - 4u_4 = -180$, i.e. $u_1 = 0.5u_2 + 30$ and $u_4 = 0.5u_2 + 45$. Substituting in $u_1 + u_4 - 4u_2 = -150$ gives $75 - 3u_2 = -150$, hence $u_2 = 75$.

$$\mathbf{u_1 = 67.5, u_2 = u_3 = 75.0, u_4 = 82.5}$$

Note: Accuracy improves as h decreases (the error is $O(h^2)$), but the number of unknowns grows like $1/h^2$, so a fine grid needs an efficient iterative solver. For non-rectangular regions, points near a curved boundary need special (unequal-step) formulas.

Key Takeaways — Unit 6

- ✓ PDE types by $B^2 - 4AC$: elliptic (< 0 , Laplace/Poisson), parabolic ($= 0$, heat), hyperbolic (> 0 , wave).
- ✓ Central differences: $u_{xx} \approx (u_{i+1,j} - 2u_{i,j} + u_{i-1,j})/h^2$, with error $O(h^2)$.
- ✓ Laplace: $u_{i,j} = 1/4(\text{sum of four neighbours})$ — the standard five-point formula; diagonal formula for initial estimates.
- ✓ Poisson: $(\text{sum of four neighbours}) - 4u_{i,j} = h^2 f(x_i, y_j)$.
- ✓ Small grids: solve the linear system directly (use symmetry); larger grids: Liebmann (Gauss–Seidel) iteration, accelerated by SOR.
- ✓ The five-point system is sparse and diagonally dominant, so the iterations converge.

— End of Unit 6 —

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